MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY

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MEETING OF 1933 NOVEMBER 10

Prof. F. J. M. Stratton, President, in the Chair.

The election by the Council of the following Fellow was duly confirmed : —

Kenneth Davies Hopkins, B.Sc., 105 Turves Green, Northfield, Birmingham.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended :—

Herbert Moore Attwell, 14 Radnor Park Avenue, Folkestone (proposed by A. Fowler) ;

Leslie Francis Ball, "Auriga," 5 Craigweil Avenue, Didsbury, Manchester (proposed by R. Barker) ;

Harold Bewick, A.M.I.C.E., 45 Percy Park, Tynemouth, Northumberland (proposed by H. Whitworth Davis) ;

William Moses Feldman, M.D., B.S., M.R.C.P., M.R.C.S., 851 Finchley Road, London, N.W. 11 (proposed by F. Robbins) ;

George Henry Gilbert, B.Sc., Launceston College, Cornwall (proposed by H. Spencer Toy) ;

Haldan Keffer Hartline, M.D., Bloomsburg, Pa., U.S.A. (proposed by R. L. Waterfield) ;

K. Mahmood Hussain, M.A., King Kothi, Hyderabad, Deccan, India (proposed by A. Amin Jung) ;

Reginald William John Payne, 64 Blackheath Road, Greenwich, S.E. 10 (proposed by D. J. R. Edney) ;

Edwin Wilbur Rice, jun., General Electric Company, Schenectady, New York, U.S.A. (proposed by C. A. Robert Lundin) ;

Walter Alexander Scott, B.Sc., 199 Rochester Way, Kidbrooke, London, S.E. 3 (proposed by L. J. Comrie) ; and

Lakshmanathirthakatte Sibaiya, M.Sc., A.Inst.P., Assistant Professor of Physics, Central College, Bangalore, S. India (proposed by K. B. Madhava).

Two hundred and fifty presents were announced as having been received since the last meeting, including :—

E. W. Brown and C. Shook, Planetary Theory (presented by Dr. Smart) ;

R. Jarry-Desloges, Observations des Surfaces Planétaires, Fascicule 8 (1921 et 1922) ;

Lady Lubbock, The Herschel Chronicle ;

H. N. Russell, Determination of Stellar Parallaxes (presented by the Carnegie Institution of Washington) ;

H. N. Russell, The Composition of the Stars (Halley Lecture, 1933) ;

W. de Sitter, The Astronomical Aspect of Relativity (presented by the University of California) ;

W. de Sitter, A Short History of the Observatory of the University at Leiden, 1633–1933 ;

W. H. Steavenson, Suns and Worlds : an Introduction to Astronomy.

Celestial Photographs.

The following have been added to the list of Celestial Photographs reproduced by the Society for sale to Fellows :—

(*N.B.*—The reference numbers given are those in the new catalogue.)

R.A.S. Ref. No.

R 1. Spectrum of Thallium under various conditions. Dingle, S. Kensington.

R 2. Spectrum of Silicon Tetrafluoride with varying degrees of ionisation. Dingle, S. Kensington.

S 6. Crossley Reflector, Lick Observatory. Ap. 36 inches, *f.l.* 17 feet 6 inches.

I 18. N.G.C. 2261. Hubble's Variable Nebula. Lowell Observatory. 1916 Mar. 2–1926 Jan. 14. 42-inch Reflector.

M 52. Spectrum of Venus and Sunlight. Lowell Observatory. 1921. 24-inch Refractor, 3-prism spectrograph.

N 16. Spectrum of μ Cassiopeiæ. Dominion Astrophysical Observatory, Victoria, B.C. 72-inch Reflector, 1-prism spectrograph.

J 89. N.G.C. 7331 (Pegasus) and Field of Nebulæ. Mount Wilson. 1916 Aug. 28. 60-inch Reflector. Exp. 6¼ h.

P 4. Spectra of Nova Pictoris. Royal Observatory, Cape. 1925 May 25–July 25. 24-inch Refractor, 4-prism spectroscope.

P 5. Spectra of Nova Pictoris. Royal Observatory, Cape. 1925 July 18–Sept. 22. 24-inch Refractor, 4-prism spectroscope.

P 6. Spectra, Nova Pictoris, Nova Aquilæ, η Carinæ. Royal Observatory, Cape. 24-inch Refractor, 4-prism spectroscope.

P 7. Spectra of Nova Pictoris. Royal Observatory, Cape. 1925 Dec. 8–1926 Mar. 2. 24-inch Refractor, 4-prism spectroscope.
P 8. Spectra of Nova Pictoris. Objective prism. Union Observatory, Johannesburg, 1926 Mar. 20–Oct. 4 and Royal Observatory, Cape, 1928.
S 10. Airy's Transit Circle. Royal Observatory, Greenwich. Ap. 8 inches, *f.l.* 12 feet.
N 17. Spectra of Field of Stars in Cygnus. Hamburg Observatory. Schmidt Coma-free Reflector, Ap. 36 cm., *f.l.* 62·5 cm., with 8° objective prism.

WORLD-RELATIONS AND THE "COSMICAL CONSTANT."

E. A. Milne, F.R.S.

1. In a recent paper,* starting from extremely general considerations, I have shown that a solution of the cosmological problem exists in which, apart from local fluctuations due to irregularities in distribution (departures from a fully smoothed-out ideal system), the velocities of the extra-galactic nebulæ relative to any one nebula are constant in time, the velocity v of each, at any one time in the experience of the observer on the reference nebula, being proportional to its relative distance r according to the law $v = r/t$, where t is the epoch reckoned from the natural zero of time. An apparent difficulty was involved in reconciling this result with the usually accepted properties of gravitation. On the one hand, the theory of Lemaître, especially as emphasised in Eddington's discussions, attributes the outward motions of the nebulæ to the repulsive effect arising out of the supposed existence of a positive "cosmical constant" λ. On the other hand, other authorities have doubted the existence of λ, or, non-committally, have supposed that it might have any value positive, zero or negative.† Kermack and McCrea ‡ have derived some of the properties of my solution by general relativity methods, *taking* $\lambda = 0$, but they found that for a non-zero density there were residual inward accelerations in their solution, and they accordingly complained § that I had "neglected gravitation."

Since the publication of my *Z.f.A.* paper I have been able to determine fully the average properties of my more general statistical solution and to integrate completely the equations of motion of the individual particles. These calculations display the statistical system of particles as breaking up into sub-systems congregating round the uniformly moving fundamental particles of the hydrodynamical system, thus giving insight into the mode by which the observed organisation of the cosmos may have developed and probably is developing. The calculations, though essentially "elementary" and employing only flat space and the times kept by observers' clocks, are lengthy, and will require time for their presentation. The present interim

* *Zs. für Astrophys.*, Bd. 6, 1, 1933.
† De Sitter has recently remarked: "The value of λ is entirely unknown" (*M.N.*, **93**, 629, 1933).
‡ *M.N.*, **93**, 519, 1933.
§ *Observatory*, **56**, 224, 1933.

note, which summarises some of the results, gives a rough sketch of the ideas which remove the apparent difficulties mentioned in the preceding paragraph, and offers a simple "explanation" of the "cosmical constant." It shows in an easily intelligible way how it comes about that it has been necessary to introduce this so-called constant, and relates the kinematic solution to Einstein's fundamental static solution.

It may be repeated that the underlying theory of the method is purely kinematical—the kinematics of motion in flat space. Accelerations arise simply as required by the single fundamental principle of the equivalence of uniformly moving particle-observers, and dynamics (which is of course a construct built up out of kinematics) is only introduced to make contact with our ordinary terminology and procedure. The same holds good of gravitation. It may be mentioned further that the "special theory" of relativity, which in my original paper was taken over as it stands, can be shown to follow from the same principle without employing the indefinable concept of the rigid body and without assuming that the particles concerned are surrounded by "empty space"; it holds good whatever the distribution of matter, for particle-observers in uniform relative motion who stand in identical relations to the surrounding distribution of matter in motion.

2. In the paper cited * I derived a relation

$$\rho_0 = \frac{-F(1)}{\frac{4}{3}\pi G t^2}, \tag{1}$$

where G is the Newtonian "constant" of gravitation, ρ_0 is the mean density of the smoothed-out universe near the observer, and $-F(1)$ was supposed to be a constant, conjectured to be of the order of unity. The arguments used were non-rigorous, as the presence of an apparent singularity at $\xi = 1$ in the acceleration function $F(\xi)$ was ignored. More detailed considerations justify a formula of the type (1); as such a formula is essential to the considerations of this note, I briefly outline the revised arguments.

The components of acceleration p_1, p_2, p_3 of a particle at x, y, z at time t moving with velocity u, v, w in the statistical system were shown to be of the form

$$p_1 = (x-ut)\frac{Y}{X}F(\xi) \equiv (x-ut)\frac{Y}{X}\left[-1 + \frac{C}{(\xi-1)^{\frac{3}{2}}\psi(\xi)}\right], \tag{2}$$

etc., where

$$X = t^2 - \frac{\Sigma x^2}{c^2}, \qquad Y = 1 - \frac{\Sigma u^2}{c^2}, \qquad Z = t - \frac{\Sigma ux}{c^2}, \qquad \xi = Z^2/XY,$$

and $\psi(\xi)$ defines the statistical distribution of particle-density

$$\frac{\psi(\xi)}{c^6 X^{\frac{3}{2}} Y^{\frac{5}{2}}} dx dy dz du dv dw. \tag{3}$$

C is a constant of integration. It will be noticed that the accelerations vanish for ever for particles for which at any one instant $u = x/t$, $v = y/t$,

* *Loc. cit.*, p. 70, equation (111).

$w = z/t$. These are of course the particles which alone occur in the hydrodynamical distribution

$$\frac{B t dx dy dz}{c^3(t^2 - \Sigma x^2/c^2)^2}, \tag{4}$$

which possesses the velocity-distribution

$$\frac{B du dv dw}{c^3(1 - \Sigma u^2/c^2)^2}. \tag{5}$$

Accordingly in the statistical system these particles move with uniform relative velocity ; each is the centre of the whole system. t is in all cases the epoch measured from the natural zero of time, and is the time kept by a clock associated with any assigned, uniformly moving, "central" particle-observer.

From the formulæ for p_1, p_2, p_3 it is possible to calculate f_r, the radial component * of acceleration of any particle, and so in turn $f_{r,0}$ the acceleration (purely radial) of a particle momentarily at rest. From the distribution function for the system, by averaging over all velocities, it is possible to determine the distribution of particle-density in the statistical system, say $n dx dy dz$. The only assumption made in carrying out the averaging process is that $\psi(\xi)$ tends to zero sufficiently rapidly as $\xi \to \infty$.

Now, astronomically, the *mass* of a particle is found by procedure equivalent to the following. Consider the particles inside a sphere of radius r, centre the origin, and examine the ratio

$$\frac{-f_{r,0}}{\frac{1}{r^2}\int_0^r 4\pi r^2 n dr}. \tag{6}$$

On the Newtonian analysis of gravitation, assuming it to hold good for small velocities at small distances r, this ratio, as $r \to 0$, tends to a definite finite limit, which is written mG, m being the mass attributed to a particle and G being independent of r. In the statistical system the distribution of particle-number completely fixes the accelerations,† so that there is no question of our being able to adjust the accelerations occurring in the presence of a given distribution of particle-number ; in other words, the description of such accelerations, which is what we mean by gravitation, is not at our disposal. But the distribution function $\psi(\xi)$ is at our disposal. Its value at $\xi = 1$ corresponds to the density at the observer. If ψ is free from singularity near $\xi = 1$, the accelerations are infinite near the observer, and no assignment of mass m is possible. If, on the other hand, the acceleration is free from singularity near $\xi = 1$, ψ must have a pronounced singularity at $\xi = 1$, the number-density is infinite, and again no assignment of

* It is possible to show that in every separating sub-system of the complete statistical system, every particle has a strictly radial acceleration relative to the associated, uniformly moving particle round which the sub-system clusters.

† Apart from the constant C, which is equivalent to an arbitrary multiplier of ψ ; in the full analysis, not here given, ψ/C comes out independent of C. That this system of accelerations describes a gravitational field is an application of the principle of equivalence, here made, however, without recourse to a Riemannian metric. The trajectories are not here geodesics in space-time.

mass-number m is possible. Nature compromises by arranging such singularities in both acceleration and distribution that mass-assignment is possible, *i.e.* that the ratio (6) above has a finite limit as $r \to 0$ which we call mG. This is an appeal to experience, which may be summarised in the proposition : Mass exists. The resulting singularity in the density corresponds to something intermediate between a continuous finite distribution and the singularity implied by the usual concept of the point-mass of finite mass.

The formula, obtained by averaging the particle-distribution (3) over all velocities, in general, may be shown to be

$$n = \frac{4\pi}{c^3 t^3 (1 - \Sigma x^2/c^2t^2)^2} \int_{(1-\Sigma x^2/c^2t^2)^{-1}}^{\infty} \frac{2\xi - 1}{(\xi - 1)^{\frac{1}{2}}} \frac{d\xi}{2\xi^{\frac{1}{2}}} \int_{\xi}^{\infty} \frac{\psi(\xi')}{2\xi'^{\frac{1}{2}}} d\xi'. \tag{7}$$

The required singularity in $\psi(\xi)$ near $\xi = 1$ comes out to be of the form

$$\psi(\xi) \sim \frac{A}{(\xi - 1)^{\frac{3}{2}}} \left(\log \frac{1}{\xi - 1} \right)^{-\frac{1}{2}} \tag{8}$$

and, near $r = 0$, n then has the asymptotic formula

$$n \sim \frac{4\pi A}{c^3 t^3} \left(2 \log \frac{ct}{r} \right)^{\frac{1}{2}}. \tag{9}$$

The acceleration has the asymptotic formula

$$f_{r,0} \sim \frac{r}{t^2} \left[-1 + \frac{C}{A} \left(2 \log \frac{ct}{r} \right)^{\frac{1}{2}} \right]. \tag{10}$$

We accordingly find that the limiting process applied to (6) gives

$$mG = -\frac{3c^3 t}{16\pi^2} \frac{C}{A^2}. \tag{11}$$

It follows that C is negative. It follows also that if we conventionally require m to be constant in time, then $G \propto t$. This means that the so-called "constant" of Newtonian gravitation should be increasing at present at the rate of about one part in two thousand million per year. Such a change would be beyond present detection, save possibly in such a phenomenon as the secular acceleration of the Moon. But it must be remembered that our result as to G applies only to mass and gravitation on the galactic scale. We have in effect applied (6) to the accelerations of particles at distances small compared with the separations of the extra-galactic nebulæ but large compared with, say, the dimensions of the solar system ; $f_{r,0}$ may be regarded as the acceleration of a particle in the presence of, and in the neighbourhood of the centre of, a given galaxy.

It is now readily shown that owing to the mildness of the singularity * in n (logarithmic only), the mean density inside the small sphere of radius r is asymptotically equal to the actual density at radius r. Call this local mean density (near the observer) ρ_0. It is what an observer would measure after

* This singularity represents in some way the observed aggregation of the material of the cosmos near the extra-galactic centres.

smoothing out the nebular concentrations. We can speak of an approximate local homogeneity as soon as the second term in the acceleration formula (10) is comparable with the first, *i.e.* as soon as r (though still small) is such that

$$\frac{C}{A}\left(2\log\frac{ct}{r}\right)^{\frac{1}{2}}$$

is comparable with unity. By (11) this requires that

$$\frac{16\pi^2 A}{3}\frac{mG}{c^3 t}\left(2\log\frac{ct}{r}\right)^{\frac{1}{2}}$$

be comparable with unity, *i.e.* by (9) that

$$\frac{4\pi}{3}nmGt^2$$

be comparable with unity. But $nm=\rho_0$, as defined. Hence the local mean density ρ_0 is given approximately by

$$\rho_0 \sim \frac{1}{\frac{4}{3}\pi G t^2}. \qquad (12)$$

3. This is the desired refinement of (1). It is equivalent, in a different notation, to the formula of general relativity for the density of a "homogeneous" universe. For example, in other units the density of a homogeneous Einstein universe of *radius of curvature* R is given by $\rho = 1/4\pi R^2$. The counterpart, in our flat space, of the radius of curvature R, is the actual radius of the sphere accessible to observation, namely, ct, and allowing for the differing mass-units (our (12) gives ρ_0 in grm. cm.$^{-3}$) the formula just quoted is identical with (12).

4. But it must be noted that ρ_0 is essentially the mean density near the observer. The observed density at epoch t increases outwards according to the hydrodynamical formula (4), in our system. The present numerical value of ρ_0 given by (12) is 10^{-27} grm. cm.$^{-3}$. If we spread the contents of our own galaxy, say $2\times 10^{11}\odot$,* over a cube of side equal to the distance of our nearest extra-galactic neighbours (say 10^6 light-years) the mean density comes out as $0{\cdot}4\times 10^{-27}$ grm. cm.$^{-3}$. Our nearest extra-galactic neighbours are supposed to contain only about $10^9\odot$, and as is well known a smaller density then results for the observed ρ than is given by (12). This is not the occasion to discuss the trustworthiness of the usual estimate of $10^9\odot$ per galaxy, but it appears to me that this is likely to be an underestimate.

5. In formula (4) the particles counted are galaxies, not the idealised ultimate particles of the statistical system. If m_1 is the mass of a nebula, the local mass-density near the observer ($x=y=z=0$) is by (4)

$$\frac{m_1 B}{c^3 t^3}. \qquad (13)$$

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* Eddington, Council Note, *M.N.*, 1928 February.

Equating this to (12) we have

$$\tfrac{4}{3}\pi m_1 B = \frac{c^3 t}{G}. \qquad (14)$$

Since B is a constant, we have once again $G \propto t$. This amounts to an independent derivation of our result of the secular dependence of G on t; for, it may be recalled, the hydrodynamical and statistical world-systems, though based on one and the same fundamental kinematical principle, were derived by independent mathematical analyses. We have thus verified their consistency, or alternatively the algebraic accuracy of our averaging process leading to (7).

6. A simple physical meaning can now be given to (14). The observable universe on the hydrodynamical model is an expanding sphere of radius ct. (The same can be shown to be true of the statistical system, though here the proof is more complicated, since the statistical system can be described for values of x, y, z lying outside the sphere $\Sigma x^2 = c^2t^2$, just as well as for values inside it; the proof depends on the integration of the trajectories of the statistical system, which shows that all *observable* trajectories start from $x=0, y=0, z=0$ at $t=0$, and that there is no interaction of any kind between particles interior to the expanding sphere in question and particles exterior to it.*) The volume of this expanding sphere at time t is $\frac{4}{3}\pi(ct)^3$. Let us fill this volume *homogeneously* with matter of density equal to the density at the observer, namely, ρ_0. The mass of this "extrapolated homogeneous universe" is by (13)

$$\frac{m_1 B}{c^3 t^3} \cdot \tfrac{4}{3}\pi(ct)^3$$

or

$$\tfrac{4}{3}\pi m_1 B.$$

By (14) this is just c^3t/G. Accordingly c^3t/G is the mass of the "extrapolated homogeneous universe," *i.e.* the mass which would be assigned to the universe by an observer who measured the local mean density near himself, and ascertained the maximum distance of observable objects (namely, ct) by means of the observed expansion, but erroneously supposed the universe to be homogeneous. The actual mass of the universe, on our models, is of course infinite. The value of c^3t/G is $2{\cdot}4 \times 10^{55}$ grm., the mass assigned to the actual universe by the theories of Lemaître and Eddington. In fact c^3t/G (apart from a numerical factor) is the mass of a homogeneous Einstein static universe; for in different units the mass of the latter is $\frac{1}{2}\pi R$, and R has for its counterpart ct.

7. I have stressed this formal algebraic similarity between the density- and mass-relations in the kinematic theory and in the general relativity theories. But the essential differences must not be ignored. In general relativity G is treated as a constant, whereas in my theory, by two distinct lines of analysis,

* A result substantially similar to this, obtained by a different method, was kindly sent to me by Prof. J. v. Neumann, of Budapest, before my own calculations had been carried out.

G is found to be proportional to t. I do not assume the existence of any "constant" of gravitation, or insert any such constant empirically into the acceleration relations; the only essential constant in my treatment is c^3t/G. A number G, independent of r, arises as a possible dynamical mode of description of the kinematic results, consequent on a choice of a singularity in ψ; G emerges then as independent of r, but proportional to t. I fully recognise the revolutionary character of this inference. It is of course a matter of convention whether we say that the mass of a particle, determined astronomically, changes with t, or whether G changes with t. I have put the burden on G.

8. We proceed now to examine that other constant of gravitation, the "cosmical constant" λ. It is known that the assumption of the existence of λ (positive) is equivalent to the assumption of a repulsive acceleration varying directly as the distance. To make contact with such a supposed effect we naturally go to large-scale phenomena, and we therefore leave now the detailed accelerations of the particles near any given galactic centre and proceed to the consideration of the nebular accelerations themselves. We therefore examine the fully smoothed-out large-scale hydrodynamical system of the theory, and see what it has to tell us about large-scale gravitation. We consider its kinematic and dynamical interpretations separately.

(1) In this system each particle moves with constant relative velocity. This may be considered as an inference from the acceleration formulæ of the statistical system, applied to the fundamental particles, as we have seen. It is far simpler, however, and more in accordance with the spirit of the kinematic treatment, to see this directly. The following new version of the proof is offered. Take a system of frames of reference moving with uniform velocity and distributed according to (4) and (5), and place a particle instantaneously at rest in each. The particle O is central in its own frame, and so unaccelerated in this frame. Let $\mathbf{f}$ be the acceleration in O's frame of any other particle P. Transform to the frame F in which P is instantaneously at rest; let $\mathbf{f}'$ be the transformed acceleration. Then P is central in F, and so $\mathbf{f}'$ is zero. Transforming back again to O's frame we see that $\mathbf{f}$ is zero. Accordingly each particle continues to move with uniform velocity, and remains at rest in its own frame. In the kinematic interpretation, then, there is nothing more to be said than that each particle undergoes zero acceleration in virtue of the structure of the complete system.

(2) Consider now the dynamical interpretation. The observer notes that the particles are in uniform velocity, but is unwilling to surrender his belief in universal Newtonian attraction derived from his observation of the accelerations of slowly moving particles in his own neighbourhood. He wishes to retain the inward accelerations he assigns to the particles as caused by the sphere of material interior to them, and to compensate such inward accelerations he introduces outward repulsions, only manifest on this the large scale. Let us see whether he can get a balance by combining Newtonian attraction and cosmic repulsion of the type introduced by Einstein.

We are only entitled to apply the Newtonian law in its usual form for particles moving with not too large velocities, and the central observer O

accordingly fixes attention on a particle P which is close * to him on the cosmic scale of distances. It is typified by a neighbouring nebula. This is acted on by (*a*) the Newtonian attraction of the whole system, (*b*) the Einstein repulsion of the whole system. The observer O calculates the Newtonian attraction as

$$-GM(r)/r^2,$$

where $M(r)$ is the mass inside $OP=r$. Consider next the Einstein repulsion due to any particle S. The Einstein repulsions due to all such particles S

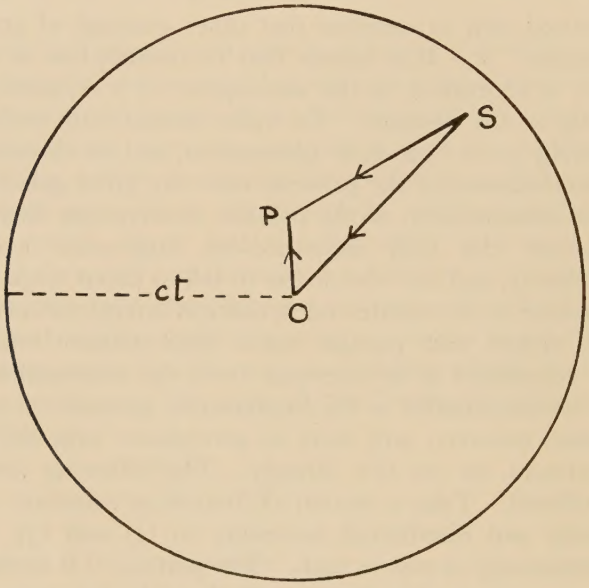

FIG. 1.

reduce at O to resultant zero, for O is the centre of the system. O, however, regards P as ex-centric and resolves the vector repulsion proportional to SP into vectors proportional to SO and OP. The vectors proportional to SO have resultant zero. The remaining vectors are all along and proportional to OP and have a sum, say, λOP.

The particles or nebulæ P will now have zero acceleration provided

$$-\frac{GM(r)}{r^2}+\lambda r=0$$

for all small r. This requires

$$\frac{M(r)}{r^3}=\frac{\lambda}{G}$$

for all small r, *i.e.* $M(r)/r^3$ must be independent of r for r small. This condition is satisfied since the smoothed-out system is locally homogeneous

* Not of course close as compared with the size of a nebula itself.

near r, as judged by O ; the variation in density with r is in fact of the order r^2/c^2t^2 or v^2/c^2. We have then from the last equation

$$\tfrac{4}{3}\pi\rho_0 = \lambda/G. \tag{15}$$

Substituting from (12) for ρ_0 we have

$$\lambda = 1/t^2. \tag{16}$$

If then λ is assigned this value, the observer will have satisfactorily reconciled Newtonian attraction, Einsteinian repulsion and the constancy of nebular velocity.

Formula (16) *is precisely Einstein's formula for his "cosmical constant"* λ *in terms of the radius of curvature of his homogeneous static universe.* His formula * is

$$\lambda = 1/R^2, \tag{16'}$$

and on replacing R by its counterpart ct and noting that in (16') the velocity of light is taken as unity, we have that (16') reduces to (16). The numerical value of the "cosmical constant" required to reconcile our system with Newtonian gravitation is identical with the value assigned to it in Eddington's discussions. By (16) λ is essentially positive and non-zero.

9. We can now see how it came about that Kermack and McCrea could accuse me of leaving out gravitation. They took $\lambda = 0$.† It was not I who left out gravitation, but they who left out the "cosmical constant." They omitted the outward repulsions, but introduced inward attractions. Consequently their solution involved inward accelerations whilst mine did not.

10. There is an apparent resemblance between the hydrodynamic solution and Einstein's homogeneous static solution in that in each case the resultant acceleration of each particle is zero ; the "field of force" reduces to zero. But in my case each velocity is constant and non-zero ; in Einstein's case all the velocities are zero. In my case the total mass or number of particles is infinite ; in Einstein's case it is finite, my "extrapolated homogeneous universe" having, however, a mass equal to the actual mass of the Einstein universe. In Einstein's case each particle is central with respect to all the others, this being arranged through the device of closed curved space ; in my case each particle is central with respect to all the others, in flat space, in virtue of the distributions of motion and of particle-number. In Einstein's case the density-distribution is homogeneous and constant in time at any point ; in my case the density-distribution is not homogeneous and the density at any point decreases as the inverse cube of the time. My solution secures the simplicity of the Einstein solution together with the added characteristic of expansion, the expansion-law involving no arbitrary constants. In Einstein's case the distribution of matter determines an absolute standard of rest, in contradiction with the spirit of relativity ; in

* This and similar formulæ are quoted from Eddington, *General Theory of Relativity*, p. 160.

† So also did Robertson, *Z.f.A.*, Bd. 7, 153, 1933. If $\lambda = 0$, then (15) is only satisfied if $\rho_0 = 0$. We thus recover the result of Robertson, Kermack and McCrea.

my case there is no absolute standard of rest, the system having no unique velocity-centroid. In Einstein's case light can make world-circuits, and from any given point its antipodal point may be seen in every direction; in my case light cannot make world-circuits and the particles at the maximum distance ct are just invisible.

11. The "cosmical constant" λ, like G, is a number independent of position but varies secularly with the time, being in fact just $1/t^2$. It is purely subjective, in two senses. First, its value depends on the epoch (by his own clock) at which the observer happens to measure it. Secondly, it corresponds to a purely subjective resolution of the acceleration *zero* into two equal and opposite components, one of which is chosen to be Newtonian attraction. In my analysis there is no action at a distance; accelerations, zero or the reverse, just "are," and the particles move in such a way that they define no preferential standard of rest or preferential view-points. The observer, however, in the neighbourhood of a galaxy, interprets these accelerations as arising from action at a distance and so is led to the Newtonian law; in the wider field of the whole cosmos, in order to retain the Newtonian action at a distance, he is then compelled to introduce another action at a distance, namely, the Einstein cosmic repulsion. On the kinematic theory here outlined both actions at a distance are simply the observer's choice of a mode of description of certain accelerations.

Eddington has attempted to infer the masses of the proton and electron from the value of the "cosmical constant." He long ago pointed out that all laws of nature are subjective – arbitrary resolutions, by the observer, of his experiences. It is therefore of some interest that the "cosmical constant" should prove to be only a subjective entity, in accordance with his philosophy. Nevertheless, there is a real sense in which Eddington's investigations concerning the cosmical constant have a firm basis. For according to the present treatment the description of the world contains a fundamental mass, $c^3t/G(2{\cdot}4 \times 10^{55}$ grm.), which may well be connected with the fundamental atomic masses, even though it is not here the mass of the universe. This constant may just as well be written $c^3/G\lambda^{\frac{1}{2}}$. In Eddington's treatment G and λ are world-constants; in the present treatment each varies secularly, but in such a way that the product $G\lambda^{\frac{1}{2}}$ remains constant.

12. It may be useful to summarise the fundamental relations between various observable quantities which are given by the kinematic theory. The relations marked with an asterisk are substantially the same as on the general relativity theories, though the interpretation is different.

(1)* From Hubble's law and the observed rate of expansion, the maximum distance of observable objects is computed. In my form, Hubble's law is $v = r/t$, and the maximum distance at epoch t is ct. The maximum observable or attainable velocity is c. The length ct is the radius of curvature of space, R, on the general relativity theories.

(2)* The mean local density near the observer, ρ_0, is connected with the rate of expansion by the formula $\rho_0 \sim 1/\frac{4}{3}\pi G t^2$.

(3) The rate of dilution of the local density in time is given by $\rho_0 \propto 1/t^3$.

(4) The rate of increase of the smoothed-out density, *in the world-map*, with distance r is given by $\rho \propto t/(t^2 - r^2/c^2)^2$.

(5) The rate of increase of the smoothed-out density, *in the world-picture*, with observed distance r_1 is given by $\rho \propto 1/t(t - 2r_1/c)^2$, where r_1 denotes the measured distance at epoch of observation t * in the camera-picture.

(6) The Newtonian "constant" of gravitation as derived from observed accelerations in and near a galaxy should be proportional to the epoch measured from the natural zero of time, $G \propto t$.

(7)* Associated with the universe is a fundamental mass c^3t/G equal to about $2{\cdot}4 \times 10^{55}$ grm., the mass of the "extrapolated homogeneous universe."

(8)* Einstein's "cosmical constant" λ is given by $\lambda = 1/t^2$.

The actual universe may be expected to contain irregularities or departures from the ideal scheme, and fluctuations or local irregularities in density-distribution may be expected to be associated with irregularities in velocity and residual accelerations. It should be possible to represent such irregularities mathematically, the principle (a kinematic one) being that all such irregularities must be consistent with the complete concealment of the existence of any standard of absolute rest or absolute velocity.

It is possible to show, though no account is here given, that the flat-space employed in deducing relations (1) to (8) is a pure construct, which can be built up solely from the temporal experiences of observers in uniform relative motion who stand in identical relations to the rest of the universe. It is not necessary to employ the concept of the rigid body in constructing this space. The whole of the special theory of relativity is deducible similarly. The constancy of the velocity of light is a pure convention which fixes the length-measures of what we usually identify as rigid bodies in terms of time-reckoning.

As regards the zero of time and the singularity sometimes called "creation," we may add :

(9) The event "creation" happened at an epoch at time t ago in our own reckoning, but time $t(1 - v^2/c^2)^{\frac{1}{2}}$ ago in the experience of an observer on a nebula moving with velocity v, reckoned from an event in his experience simultaneous (to us) with the event in our experience, we label as t. By "an event simultaneous with t" I mean an event E on P such that a light-signal despatched by us at time t_1 so as to arrive at P at the event E and immediately returned arrives back in our experience at time t_2, where $t = \frac{1}{2}(t_1 + t_2)$.

13. As an addendum I may mention that insight into the significance of the fundamental density-formula $\rho_0 \sim 1/\frac{4}{3}\pi G t^2$ may be gained as follows: If a given nebula is escaping from us so as never to return, it must have a velocity v_0 exceeding the Newtonian velocity of escape, adopting for simplicity the crude Newtonian law of gravitation. And if it once possesses such a velocity it will always possess it, for all nebulæ outside it escape faster, and the total mass interior to it, judged by ourselves considered as central, remains constant. If then $\frac{1}{2}v_0^2 > GM(r)/r$,

* *M.N.*, 93, 674, 1933.

inserting Hubble's law in the form $v_0 = r/t$ we have

$$\frac{1}{2Gt^2} > \frac{M(r)}{r^3} = \tfrac{4}{3}\pi\rho_0.$$

Hence *

$$\rho_0 < 1/\tfrac{8}{3}\pi Gt^2.$$

Comparison with (12) shows that ρ_0 is of such an order of magnitude that v_0 is of the order of the parabolic velocity. This crude calculation of course ignores what we are accustomed to call the cosmical repulsion, but it shows that it is roughly immaterial, in ascertaining the condition of escape, whether we assign a parabolic velocity or introduce a cosmical repulsion of such a size as to give zero acceleration. It is also immediately obvious that with negligible accelerations $\rho_0 \propto 1/t^3$, for the radius of the sphere through which the material of a galaxy must be evened-out, increases as t. Thus the proportionality $G \propto t$, obtained by equating the two values of ρ_0 obtained by the two different lines of argument, is almost intuitive.

THE SOLAR CHROMOSPHERE.

S. Chandrasekhar, Ph.D.

§ 1. *Introduction.*—The problem of the chromosphere is, like the problem associated with any atmosphere, to determine its physical structure, *i.e.* to explain on an adequate physical basis its observed density distribution. The chromosphere, however, presents a great difficulty at a much earlier stage, and the fundamental problem had been for long merely the question relating to its enormous extensions, as it was recognised quite early that on the basis of our ordinary ideas of statistical equilibrium we cannot have on the Sun more than a hundred kilometres of visible atmosphere while the actual heights of the chromospheres are nearer ten thousand kilometres.

But this difficulty was overcome when Milne † developed in connection with the calcium chromosphere the idea of support by monochromatic radiative equilibrium. The underlying physical ideas are well known, but it may be recalled that the law of the density distribution predicted by the theory depended very much on whether the atmosphere is a "fully supported" one or only a "partially supported" one, which simply means as to whether the *very high-level* chromospheric atoms are completely supported by radiation-pressure or by the joint working of the radiation-pressure together with a small residual limiting pressure gradient. Actually it was found that

* The cosmical repulsion permits the larger value of ρ_0, namely, $\rho_0 \sim 1/\frac{4}{3}\pi Gt^2$.

† E. A. Milne, (1) *M.N.*, 84, 354, 1924; (2) 85, 111, 1924; (3), (4) 86, 459, 578, 1926. For convenient summaries see R. H. Fowler, *Statistical Mechanics* (Cambridge), pp. 397–408; E. A. Milne, *Handbuch der Astrophysik*, Band iii/1, pp. 173–182; and A. S. Eddington, *Internal Constitution of Stars*, pp. 362–369.

in a fully supported atmosphere we have a very slow decrease of density, but if we removed from the radiation-pressure even so much as 1 per cent. of the burden of supporting the high-level atoms, then the structure would simply collapse on to the lowest levels. The chromospheric equilibrium then appeared to be an extremely delicate one, the least departure from exact balance being followed by catastrophic consequences. This delicate poise of the chromosphere led to many interesting consequences, the two most important being (i) the possibility of the emission of high-speed particles by being continually accelerated away as a consequence of the Doppler effect and the increasing flux of radiation to which the atom gets exposed, and (ii) the possibility of a general explanation of the forms and motions of the "spot" and "erruptive prominences," the lack of radiation-pressure near sunspots leading to "attraction" and the excess of it near faculæ leading to "repulsion." The first of the above effects was discovered by Milne, and the second has been the subject-matter of the investigations of the late S. R. Pike.*

§ 2. *Difficulties of the Theory and a Criticism of the Alternative Theory based on Ideas of "Turbulence."*—As the theory based on monochromatic radiative equilibrium was successful, at least to the extent of providing an adequate physical mechanism which explains the very large extension of the calcium chromosphere, it was necessary to examine further as to whether the theory and the observations agreed in the finer details regarding the march of the density distribution, and also as to whether Milne's ideas, developed in connection with the calcium chromosphere, could be applied to other cases as well—to the hydrogen chromosphere, for instance. Menzel,† in his work on the *Solar Chromosphere*, has come to the conclusion that Milne's theory, in the form in which it stands, fails under these more general tests. The work of Pannekoek and Minnaert ‡ on the hydrogen chromosphere and the analysis based on their results by McCrea § also lend evidence in favour of Menzel's conclusions.

The work of Pannekoek, Minnaert and Menzel has established that to a first very rough approximation the chromosphere in its density distribution mimics an isothermal atmosphere at a temperature at which the chromospheric atoms would be moving about with thermal velocities of the order of 15 km./sec.$^{-1}$. Menzel and McCrea further adduce evidence based on Doppler broadening measurements that these velocities are *real*.

These facts appear at first sight to be so much in contradiction with the consequences of support by monochromatic radiative equilibrium that Menzel and McCrea have abandoned this theory altogether and wish to introduce fundamentally "new ideas." These attempts (at least in the form in which they have been presented) are not at all logical deductions from well-defined premises, but are merely methods to define in some *ad hoc*

* S. R. Pike, (1) *M.N.*, **87**, 56, 1926 ; (2) **88**, 3, 1927 ; (3) **88**, 635, 1928.

† D. H. Menzel, (1) *Lick O.P.*, **17** ; also (2) *M.N.*, **91**, 628, 1931.

‡ A. Pannekoek and M. Minnaert, *Verh. d. Kon. Akad. Amsterdam*, xiii, No. 5, 1928 ; also Davidson, Minnaert, Ornstein and Stratton, *M.N.*, **88**, 536, 1928.

§ W. H. McCrea, (1) *M.N.*, **89**, 483, 1929 ; (2) **89**, 718, 1929.

manner a temperature *parameter* T^* of the order of 20,000 degrees or more for the chromospheric atoms and then to assert that the density distribution must be the same as an isothermal atmosphere at the *thermodynamic* temperature T^*. They argue in the following manner :—

In an isothermal atmosphere the density law is

$$\rho = \rho_0 e^{-\frac{mgH}{kT}},$$

where m is the mass of the atom, g the value of gravity and H the height. The numerator mgH in the exponent of e represents the potential energy of an atom of mass m in a gravitational field of intensity g at height H above the surface. The denominator kT is equal to 2/3 the average kinetic energy of the moving atom. "If we now consider *groups* of atoms in *turbulent motion*, the individual velocities of the groups approximating to a Maxwellian distribution with an average velocity C, then since the potential energy would be unaffected we should expect the density distribution of such an atmosphere to follow the law" (Menzel (1)) :

$$\rho = \rho_0 e^{-\frac{mgH}{kT+\frac{1}{3}mC^2}} = e^{-\frac{mgH}{kT^*}}.$$

By *defining* sufficiently large velocities C we can define arbitrarily large values for the temperature parameter T^*. Menzel and McCrea *deduce* values for C to fit the observed law of density distribution.

There are three main objections against the theory based on these ideas of turbulence. *Firstly*, the assumption that the groups of atoms can be distributed according to Maxwell's law appears to be physically inadmissible, for a Maxwellian distribution implies that the various groups of atoms exchange momenta ; how then do groups exchange momenta without the individual atoms taking part in the "collision processes"? * *Secondly*, even if the type of equilibrium suggested was actually realised it would not attribute to the chromosphere that degree of delicate balance which is necessary for us to attribute to it on other grounds. *Thirdly*, no theoretical justification is provided as to why C should precisely be of the order of 15 km./sec.$^{-1}$, but a really adequate theory must certainly explain as to why kinetic velocities of just this order are encountered in the chromosphere.

These objections are enough to show the untenability of these ideas based on "turbulence," and it would appear more profitable to examine in greater detail the possibilities of the support by radiation-pressure, at least for this reason that it does not seem to be generally recognised that there are in fact other possibilities.

§ 3. *Monochromatic Radiative Equilibrium over a Non-uniformly Radiating Surface.*—Now, the Schuster idealisation of the existence of a definite radiating surface naturally underlies any theory of the chromosphere,† and

* I am indebted to Professor Milne for this remark.

† The same Schuster idealisation underlies also the determination of the "number of atoms per cm.^{2} above the photosphere" (by the method of Stewart and Unsöld by a simple application of Schuster's formula for the residual intensity), but here the radiating surface is placed *below* the reversing layers. In the theory

if necessary we can also attribute to the radiating surface additional characteristics such as the periodic variations in the emergent flux of radiation as are revealed by direct observation. It is a fact of observation that the emergent flux of radiation *in* the chromospheric lines is not constant but varies from point to point on the surface of the Sun and in fact oscillates about a certain mean value specified by the residual intensity in the line. Examinations of the spectroheliograms in monochromatic light indicate that an allowance of a maximum variation of the emergent flux by a factor 2 to 3 * over an average distance of about 10,000 kilometres is a fair estimate. It would be worth while, then, to examine if the non-uniformity of the radiating surface when incorporated into the theory of support by monochromatic radiation would be sufficient to remove the difficulties which the theory at present encounters.

If we assume that the mean flux just corresponds to "full support," then the variation of the flux over the surface would accelerate the chromospheric atoms, and in a hydrodynamically steady state the velocities of the chromospheric atoms must, from dimensional considerations alone, be of the order of $\sqrt{\lambda g}$, where λ is "wave-length" of the flux periodicity over the surface and g the solar value of gravity. If $\lambda = 20{,}000$ kilometres, then $\sqrt{\lambda g}$ is $73{\cdot}6$ km./sec.$^{-1}$. There will be other numerical factors coming in, but it is clear that it is reasonable to hope that the chromospheric atoms will have on this theory such kinetic velocities as are demanded by observation.

§ 4. *General Outline of the Method.*—It is convenient to regard the flux emitted by the radiating surface as being periodic in the X and Y directions (the Z direction is the vertical) with certain specified wave-lengths λ_x and λ_y. That is, we assume that

$$I(x, y) = I_0 + I_1(x, y), \tag{1}$$

and

$$I_1(x + n\lambda_x, y + m\lambda_y) = I_1(x, y), \tag{1'}$$

where $I(x, y)$ is the intensity of the emitted radiation at the point (x, y), which is assumed to be locally isotropic in the hemisphere ; † I_0 is a constant and m and n are arbitrary integers. We shall further assume that the atom would be in equilibrium when the flux of radiation is that corresponding to I_0. Then

$$\tfrac{1}{4}B_{1\to 2}I_0 = mg, \tag{2}$$

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of the chromosphere, on the other hand, the radiating surface is placed *above* the reversing layers, and it is the flux of radiation corresponding to the residual intensity which is responsible for supporting the chromospheric atoms. Thus the "number of atoms" determined by Stewart's method has nothing to do with the number of atoms in a column of the chromosphere *above* the reversing layers. The non-recognition of this fact has led to much confused statements in the literature.

* The allowance of a factor 2 appears to be a safe *lower* limit. See in this connection G. E. Hale and F. Ellerman, *Proc. Nat. Acad. Sci.*, 1, 102, 1916. I am indebted to Professor H. H. Plaskett for this reference and also for valuable information on the observational side of these questions.

† "Darkening effects" can be neglected in a first survey.

where $B_{1\to 2}$ is the appropriate Einstein *absorption* coefficient,* and g the solar value of gravity. Hence it is only the component $I_1(x, y)$ which would give rise to a "field of force," † and we need the components of the force at an arbitrary point x, y, z. If we denote these by H_x, H_y and H_z, then clearly

$$H_x = \frac{zB_{1\to 2}}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_1(x'+x, y'+y)\frac{x'dx'dy'}{(x'^2+y'^2+z^2)^2}, \tag{3}$$

$$H_y = \frac{zB_{1\to 2}}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_1(x'+x, y'+y)\frac{y'dx'dy'}{(x'^2+y'^2+z^2)^2}, \tag{4}$$

$$H_z = \frac{z^2B_{1\to 2}}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_1(x'+x, y'+y)\frac{dx'dy'}{(x'^2+y'^2+z^2)^2}. \tag{5}$$

This field of force (H_x, H_y, H_z) will yield a system of dynamical equations of motion which will have, in general, periodic as well as aperiodic solutions. It is clear that the permanence of the chromosphere requires the atoms in the chromosphere to describe *periodic* trajectories.‡ We should therefore have the components of the velocity u, v, w to be all periodic functions in x and y. In determining the density distribution associated with this derived field of velocities we must remember that we are treating an atmosphere which is not in thermodynamical equilibrium and that as a consequence we cannot introduce any conception of temperature. The problem is essentially a *hydrodynamical* one and we are provided with only one partial differential equation for ρ (the equation of continuity) as the condition for the maintenance of a *steady state.* We should have in fact

$$u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}+w\frac{\partial\rho}{\partial z}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0. \tag{6}$$

This equation will not specify the density distribution uniquely, but only to the extent of an arbitrary function. But the boundary conditions which a density distribution function of an atmosphere will have to satisfy will remove this indeterminateness.

§ 5. In this paper I shall confine myself to working out the details of the equilibrium for the case

$$I_1(x, y) = I_1 \sin\frac{2\pi}{\lambda}x, \tag{7}$$

where I_1 is a constant. With this assumption the problem admits of exact mathematical solution and we can, with a start made this way, hope to obtain insight into the more general problem.§

* $B_{1\to 2}$ as here defined is $h\nu_0/c$ times the usually defined Einstein *probability* coefficient.

† It may be mentioned that this field-force cannot be derived from a "potential."

‡ The aperiodic trajectories are likely to be associated with some types of prominences (see §§ 12, 17).

§ It is easy to write down the equations of motion for more general forms for $I_1(x, y)$ than (7), but as a full mathematical treatment even for the case

$$I_1(x, y) = I_1 \sin\frac{2\pi}{\lambda}x \sin\frac{2\pi}{\lambda}y, \tag{7'}$$

requires considerable numerical work, I shall reserve these refinements for a future occasion.

§ 6. *The Dynamical Equations of Motion.*—H_y is naturally zero. By (3), (5) and (7) we have

$$H_x = \frac{zB_{1\to 2}I_1}{\pi} \cos\frac{2\pi}{\lambda}x \int_0^\infty x' \sin\frac{2\pi}{\lambda}x' \,.\, dx' \int_0^\infty \frac{dy'}{(x'^2+y'^2+z^2)^2}, \tag{8}$$

$$H_z = \frac{z^2B_{1\to 2}I_1}{\pi} \sin\frac{2\pi}{\lambda}x \int_0^\infty \cos\frac{2\pi}{\lambda}x' \,.\, dx' \int_0^\infty \frac{dy'}{(x'^2+y'^2+z^2)^2}, \tag{8'}$$

which after some reductions can be expressed in the forms

$$\left.\begin{aligned} H_x &= \frac{B_{1\to 2}I_1}{4} \cos\xi \,.\, \zeta K_0(\zeta), \\ H_z &= \frac{B_{1\to 2}I_1}{4} \sin\xi \,.\, \zeta K_1(\zeta), \end{aligned}\right\} \tag{9}$$

where

$$\xi = \frac{2\pi}{\lambda}x\,; \qquad \zeta = \frac{2\pi}{\lambda}z, \tag{10}$$

and

$$K_\nu(\zeta) = \frac{\Gamma(\nu+\frac{1}{2})(2\zeta)^\nu}{\Gamma(\frac{1}{2})} \int_0^\infty \frac{\cos\xi d\xi}{(\xi^2+\zeta^2)^{\nu+\frac{1}{2}}}. \tag{11}$$

From (9), using the relation (2) to eliminate $B_{1\to 2}$, we derive our *fundamental set of dynamical equations* :

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} &= a\cos\xi \,.\, \zeta K_0(\zeta), \\ \frac{d^2\zeta}{dt^2} &= a\sin\xi \,.\, \zeta K_1(\zeta), \end{aligned}\right\} \tag{12}$$

where

$$a = \frac{2\pi}{\lambda} \,.\, rg\,; \qquad r = \frac{I_1}{I_0}. \tag{12'}$$

The functions $\zeta K_0(\zeta)$ and $\zeta K_1(\zeta)$ are tabulated in Table III of the Appendix.

§ 7. The functions K_ν introduced in the previous section are related to the Hankel functions with imaginary arguments : *

$$K_\nu(\zeta) = \tfrac{1}{2}\pi i e^{\frac{1}{2}\nu\pi i} H_\nu^{(1)}(i\zeta). \tag{13}$$

There are a number of recurrence relations which these K_ν satisfy (see Watson, *Bessel Functions*, p. 79), but the relations we most often need are the following (easily verified from the definition (11)) :—

$$K_0'(\zeta) = -K_1(\zeta), \tag{14}$$

$$\zeta K_1'(\zeta) + K_1(\zeta) = -\zeta K_0(\zeta), \tag{15}$$

where a dash denotes differentiation with respect to ζ. Further, for ζ large we have the asymptotic relation :

$$K_\nu(\zeta) = \left(\frac{\pi}{2\zeta}\right)^{\frac{1}{2}} e^{-\zeta}\left[1 + \frac{4\nu^2-1^2}{1\,!\,8\zeta} + \frac{(4\nu^2-1^2)(4\nu^2-3^2)}{2\,!\,(8\zeta)^2} + \ldots\right]. \tag{16}$$

* See G. N. Watson, *Bessel Functions* (Cambridge), 1922, p. 78.

Because of this exponential factor we see that the accelerations produced by the radiating surface tend to zero for ζ large, and one immediate consequence of this result is that the " height " of the chromosphere must be of the same order as the wave-length λ. This deduction is confirmed by observation.

Now the vertical acceleration is proportional to $\zeta K_1(\zeta)$, which is monotonic decreasing, taking the value unity for $\zeta = 0$. On the other hand, the horizontal acceleration which is proportional to $\zeta K_0(\zeta)$ passes through a *maximum*, $\zeta K_0(\zeta)$ taking the value 0 for both $\zeta = 0$ and $\zeta = \infty$. The maximum occurs at a value for ζ which is the root of the equation

$$\zeta K_0'(\zeta) + K_0(\zeta) = 0,$$

or

$$\zeta K_1(\zeta) = K_0(\zeta). \tag{17}$$

The solution of (16) is found numerically to be

$$\zeta = 0{\cdot}595. \tag{17'}$$

The occurrence of this maximum confirms an earlier result of Pike (2). He states that "a spot has influence high up above the chromosphere whilst affecting the lower regions very little."

§ 8. *An Integral of the Equations of Motion.*—The equations of motion (12) possess an integral which can be obtained as follows: Using (15), we derive

$$\frac{d}{dt}\left(\frac{d\xi}{dt}\frac{d\zeta}{dt}\right) = a\cos\xi \,.\, \zeta K_0(\zeta)\frac{d\zeta}{dt} + a\sin\xi \,.\, \zeta K_1(\zeta)\frac{d\xi}{dt} = -\frac{d}{dt}[a\cos\xi \,.\, \zeta K_1(\zeta)]. \tag{18}$$

Integrating (18) we have that

$$\frac{d\xi}{dt}\cdot\frac{d\zeta}{dt} + a\cos\xi \,.\, \zeta K_1(\zeta) = \text{constant}. \tag{19}$$

§ 9. *A Necessary and Sufficient Condition for the Existence of Periodic Solutions.*—As stated in § 4, we have to seek solutions of (12) which yield for $\dot{\xi}$, $\dot{\zeta}$ * and ζ (and therefore also for $\zeta K_1(\zeta)$) periodic functions in ξ, *i.e.* they should each be capable of being expanded in a Fourier series. We shall assume that

$$\dot{\xi} = \Sigma a_n e^{in\xi}\,; \qquad \dot{\zeta} = \Sigma b_n e^{in\xi}, \tag{20}$$

and that further

$$\zeta K_1(\zeta) = \Sigma c_n e^{in\xi}. \tag{21}$$

In (20) and (21) (and also in the equations which follow) the summation over n extends from $+\infty$ to $-\infty$. By (20) we have

$$\ddot{\xi} = i[\Sigma a_n n e^{in\xi}]\dot{\xi}, \tag{22}$$

$$\ddot{\zeta} = i[\Sigma b_n n e^{in\xi}]\dot{\xi}, \tag{23}$$

and by (15)

$$\zeta K_0(\zeta)\dot{\zeta} = -i[\Sigma c_n n e^{in\xi}]\dot{\xi}. \tag{24}$$

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* A dot denotes differentiation with respect to the time t.

Substituting (20), (22) and (24) in the first of the equations (12) and equating the coefficients of $e^{in\xi}$ we have

$$\sum_{r=0}^{n} a_r r b_{n-r} = -\frac{a}{2}[(n-1)c_{n-1} + (n+1)c_{n+1}]. \tag{25}$$

Similarly from (20), (21), (23) and the second of the equations (12) we obtain

$$\sum_{r=0}^{n} a_r (n-r) b_{n-r} = -\frac{a}{2}[c_{n-1} - c_{n+1}]. \tag{26}$$

Adding (25) and (26) we have

$$\sum_{r=0}^{n} a_r b_{n-r} = -\frac{a}{2}[c_{n-1} + c_{n+1}], \tag{27}$$

which by (20) and (21) is equivalent to the relation

$$\dot{\xi}\dot{\zeta} = -a \cos \xi \,.\, \zeta K_1(\zeta). \tag{28}$$

From (28) we deduce that *the necessary and sufficient condition that an atom starting at a point* $(0, \zeta_0)$ *on the* ζ*-axis should describe a periodic trajectory is that*

$$\dot{\xi}_0\dot{\zeta}_0 = -a\zeta_0 K_1(\zeta_0). \tag{29}$$

§ 10. *The Solution of the Equations of Motion.*—It is seen at once that the equations (12) combined with the condition (28) possess the following *first integrals* :—

$$\frac{d\xi}{dt} = \alpha\zeta K_1(\zeta), \tag{30}$$

$$\frac{d\zeta}{dt} = -\beta \cos \xi, \tag{31}$$

where α and β are positive constants such that

$$\alpha\beta = a. \tag{32}$$

It is easy to prove (by an analysis similar to that in § 9) that (30) and (31) are the only solutions which satisfy the further *necessary* condition that $\dot{\xi}$, $\dot{\zeta}$ and ζ are also periodic functions in time. We shall, however, omit the details of this proof.

From (30) and (31) we obtain that

$$-\beta \cos \xi d\xi = \alpha\zeta K_1(\zeta) d\zeta, \tag{33}$$

which on integration yields the following *equation for the trajectory* :—

$$\sin \xi = \frac{\alpha}{\beta}\int_{\zeta}^{\zeta_0} \zeta K_1(\zeta) d\zeta, \tag{34}$$

where ζ_0 is the point at which the trajectory intersects the ζ-axis. Equations (30), (31) and (34) express the complete solution of the dynamical problem.

§ 11. *A Discussion of the Form of the Trajectories.*—We shall introduce the quantity $Q(\zeta_1, \zeta_2)$ defined by

$$Q(\zeta_1, \zeta_2) = \int_{\zeta_2}^{\zeta_1} \zeta K_1(\zeta) d\zeta. \tag{35}$$

The equation of the trajectory can then be expressed by

$$\sin \xi = \frac{\alpha}{\beta} Q(\zeta_0, \zeta). \tag{36}$$

The values of $Q(\zeta, 0)$ for different values of ζ are tabulated in the last column of Table III in the Appendix. Further, we have the result easily obtained (see equation (56)) :

$$Q(\infty, 0) = \frac{\pi}{2}. \tag{37}$$

Now, it is clear that *the atoms* (that are *in* the chromosphere) *should describe only such trajectories that they do not either strike the radiating surface or go off to infinity.* This will impose certain restrictions on the values of α and β.

Let ζ_1 be such that

$$Q(\zeta_1, 0) = Q(\infty, \zeta_1) = \tfrac{1}{2} Q(\infty, 0) = \frac{\pi}{4}. \tag{38}$$

Numerically ζ_1 is found to be 0·942. Consider firstly trajectories such that $\zeta_0 \leqslant \zeta_1$. ($\zeta_0$ is the point at which the trajectory intersects the ζ-axis.) The condition that the trajectory should have no intersections with the ξ-axis yields the inequality

$$\frac{\alpha}{\beta} Q(\zeta_0, 0) \geqslant 1, \tag{39}$$

or since $\alpha\beta = a$ we have

$$\alpha_{\min} = \sqrt{\frac{a}{Q(\zeta_0, 0)}}; \qquad \beta_{\max} = \sqrt{Q(\zeta_0, 0) \,.\, a}. \tag{40}$$

Actually the fact that $\alpha_{\min} \to \infty$ as $\zeta_0 \to 0$ need not trouble us since physically the condition that a chromospheric atom should not hit the radiating surface merely means that in the physical problem a chromospheric atom descending into the reversing layers gets into regions where approximations valid for the greater part of the chromosphere are no longer true and the mathematical singularity at $\zeta_0 = 0$ is merely a consequence of a rigorously physically inadmissible idealisation of the existence of a definite radiating surface which underlies the mathematical formulation.

Again since $\zeta_0 \leqslant \zeta_1$ the condition (39) automatically restricts the atom from not going off to infinity. Actually when $\zeta_0 = \zeta_1$ the atom which just grazes the radiating surface goes off to infinity but with a zero limiting velocity.

Now, let $\zeta_0 > \zeta_1$. The condition now that the atom does not go off to infinity yields instead of (39) that

$$\frac{\alpha}{\beta} Q(\infty, \zeta_0) > 1, \tag{41}$$

or

$$\alpha_{\min} = \sqrt{\frac{a}{Q(\infty, \zeta_0)}}; \qquad \beta_{\max} = \sqrt{Q(\infty, \zeta_0) \cdot a}. \tag{42}$$

The inequality (41) also restrains the atom from hitting the radiating surface.

If, however, the condition (41) is not satisfied, then the atom goes off to infinity with a certain *limiting vertical velocity* * given by

$$[\dot{\zeta}]_\infty = \beta \cos\left[\sin^{-1}\frac{a}{\beta}Q(\infty, \zeta_0)\right]. \tag{43}$$

If $aQ(\infty, \zeta_0)/\beta$ is unity, then the atom is at rest at infinity.

The values of $\alpha_{\min}$, $\beta_{\max}$ and the actual *minimum* horizontal velocities † specified by $\alpha_{\min} \cdot \zeta_0 K_1(\zeta_0)$ are tabulated in Table I.

TABLE I

ζ_0	$\alpha_{\min} \cdot a^{-1/2}$	$\beta_{\max} \cdot a^{-1/2}$	$[a\zeta_0 K_0(\zeta_0)]_{\min} a^{-1/2}$
0	∞	0	∞
0·5	1·4667	0·6818	1·2148
1·0	1·1552	0·8656	0·6953
1·5	1·4186	0·7049	0·5903
2·0	1·7543	0·5700	0·4907
2·5	2·1807	0·4586	0·4028
3·0	2·721	0·3675	0·3278
3·5	3·405	0·2937	0·2651
4·0	4·272	0·2341	0·2134
4·5	5·371	0·1862	0·1711
∞	∞	0	0

To see precisely what these values correspond to in actual velocity measures we shall take $\lambda = 20{,}000$ km. and $r = 1/3$ (this corresponds to a maximum variation of flux by a factor 2 which, as has been pointed out, is a lower limit. $r = 1/2$ would appear to be more appropriate, but we shall assume that $r = 1/3$ to be on the safer side). On these assumptions $\dot{\xi}$ or $\dot{\zeta}$ equal to $\sqrt{a}$ corresponds to the velocity components u or w given by the *fundamental unit of velocity* V specified by

$$V = \sqrt{rg \cdot \frac{\lambda}{2\pi}} = 17{\cdot}0 \text{ km./sec.}^{-1}. \tag{44}$$

We therefore see that the theory predicts just the order of kinetic velocities which observations indicate and which the advocates of turbulence have been looking for. We, however, notice that in contrast to McCrea and Menzel, who *deduced* values for their "C," we are now able to *predict* the

* The horizontal velocity is zero.

† The actual velocities are much greater. See § 14, equation (59).

correct order for the kinetic velocities on reasonable assumptions regarding the surface of the Sun and on the basis of a logically worked-out theory.

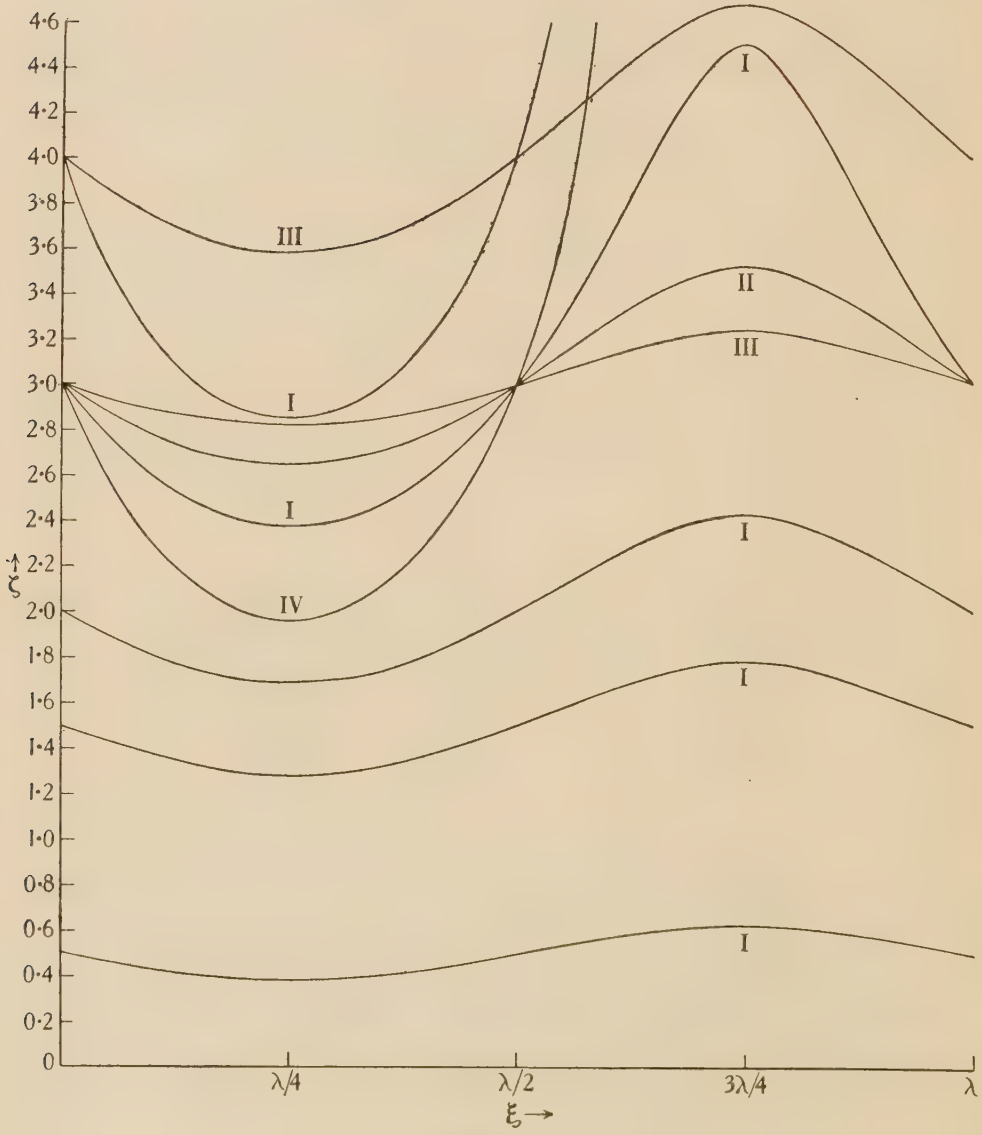

Figure 1.

Trajectories of the Chromospheric Atoms
I→α/β=10; II→α/β=20; III→α/β=40; IV→α/β=5

In fig. 1 we have drawn as illustration two different sets of trajectories.* (1) Trajectories with different ζ_0 but with a given value for $\alpha/\beta(=10)$.

* Different pairs of values (ξ, ζ) on the trajectories were calculated, using the table of values $Q(\zeta, 0)$ tabulated in the Appendix. It has not been thought worth while to give the details of the solutions.

The atoms describing these trajectories all have the same *downward* vertical velocity when crossing the ζ-axis. We notice that as we go higher up the trajectories become more and more "fluctuating," and in fact the atom starting with this value for α/β at $\zeta_0 = 4{\cdot}0$ actually goes off to infinity. The limiting value for $\dot{\zeta}$ as given by (43) for this trajectory is found to be

$$[\dot{\zeta}]_\infty = 0{\cdot}8365\beta = 0{\cdot}2645\sqrt{a}. \tag{43'}$$

(2) Trajectories of atoms all starting at the same height ($\zeta_0 = 3{\cdot}0$) but with decreasing downward vertical velocities corresponding to $\alpha/\beta = 5$, 10, 20 and 40. The atom characterised by $\alpha/\beta = 5$ actually goes off to infinity, and the limiting value in this case for $\dot{\zeta}$ is found to be

$$[\dot{\zeta}]_\infty = 0{\cdot}7375\beta = 0{\cdot}3298\sqrt{a}. \tag{43''}$$

We have also drawn a trajectory starting at $\zeta_0 = 4$ with $\alpha/\beta = 40$.

§ 12. *An Aperiodic Solution of the Equations of Motion and the Possibility of the Emission of moderately High-speed Atoms.*—Before proceeding to the discussion of the density distribution in the chromosphere we shall now, while discussing the equations of motion, show that our theory predicts the possibility of a continual emission of moderately high-speed particles. We have already noticed a class of trajectories which go to infinity with certain limiting velocities, but these velocities are not very large. On the other hand there exist *aperiodic* solutions of the equations of motion, along the trajectories corresponding to which the atom will have quite large velocities. Thus consider an atom starting from the point $(0, \lambda/4)$ with zero horizontal velocity. By (12) the trajectory is clearly a straight line parallel to the ζ-axis. The equations of motion (12) reduce to the one equation

$$\frac{d^2\zeta}{dt^2} = a\zeta K_1(\zeta), \tag{45}$$

from which we deduce that

$$\dot{\zeta}^2 = \dot{\zeta}_0{}^2 + 2a\int_0^\zeta \zeta K_1(\zeta)\,d\zeta. \tag{46}$$

By (37) we have

$$\dot{\zeta}_\infty{}^2 = \dot{\zeta}_0{}^2 + a\pi. \tag{47}$$

An atom starting with even a zero vertical velocity will end with a limiting vertical velocity specified by

$$\dot{\zeta}_\infty = \sqrt{rg \cdot \frac{\lambda}{2}} = 30{\cdot}2 \text{ km./sec.}^{-1}. \tag{48}$$

This possibility of this emission of *moderately high* speed atoms is of entirely different origin from the possibility of the emission of *very high* speed atoms pointed out by Milne (3).

§ 13. *The Density Distribution in the Chromosphere.*—In §§ 9, 10 and 11 the complete solution to the dynamical problem was obtained. It was found

that an atom starting at a point $(\zeta_0, 0)$ can, consistent with the equations of motion and with the additional restriction that it should describe a periodic trajectory, have a range of values for α. We shall now impose the further condition that *the time derivatives* $\dot{\xi}$ *and* $\dot{\zeta}$ *are to be unique functions in the whole* ξ-ζ *plane.* We should not expect that this restriction would be rigorously satisfied in an actual chromosphere. Far more is it likely that the uniqueness of the velocity components is only very approximately realised in an actual chromosphere, but in assuming a rigorous uniqueness we shall be avoiding all the difficulties associated with the hydrodynamics of a fluid in which the velocity components are many valued functions of position.

It is easy to see that the condition for the uniqueness of $\dot{\xi}$ and $\dot{\zeta}$ restricts α/β *and therefore also* α *and* β *separately to be constants of the atmosphere.* With α/β constant the hydrodynamical problem is quite elementary. The equation of continuity (6) in the ξ, ζ variables takes the form

$$\dot{\xi}\frac{\partial \rho}{\partial \xi}+\dot{\zeta}\frac{\partial \rho}{\partial \zeta}+\rho\left(\frac{\partial \dot{\xi}}{\partial \xi}+\frac{\partial \dot{\zeta}}{\partial \zeta}\right)=0, \tag{49}$$

which by (30) and (31) reduces to

$$\alpha\zeta K_1(\zeta)\frac{\partial \rho}{\partial \xi}=\beta\cos\xi\frac{\partial \rho}{\partial \zeta}. \tag{50}$$

The solution of (50) is

$$\rho=\Phi\left\{\sin\xi+\frac{\alpha}{\beta}\int_0^{\zeta}\zeta K_1(\zeta)d\zeta\right\}, \tag{51}$$

where $\Phi(x)$ is an arbitrary function in the argument x.

§ 14. From (51) we see that the existence of a definite radiating surface emitting radiation with a flux which varies periodically over the surface, leads to a density distribution which necessitates the density at the base of the chromosphere to be a periodic function as well. In a first approximation it would be legitimate to assume that the density at the base of the chromosphere varies simply harmonically. This would mean that the function Φ is merely a multiple of the argument specified plus another constant. (A justification for this assumption is provided in § 15, where the conception of a "natural boundary" of a chromosphere is introduced.)

We do not, however, know what value we are to assign to α/β. We shall show in the following section that the law of density variation at the base of the chromosphere will itself control to some extent the precise value of α/β which will characterise the *normal* chromosphere. Since, however, the chromosphere extends (as a fact of observation) up to at least $4{\cdot}5\zeta$, *we shall assume provisionally that* $\alpha/\beta(=\alpha^2/a)$ *is* 40, for then the trajectories at least up to $\zeta_0=4{\cdot}0$ are not wildly fluctuating, and it is only when ζ_0 approaches 5 that the atoms characterised with this value for α/β describe trajectories which go off to infinity.* If we assign a value 40 we see that the second term in $\{\}$ in (51) will be very large compared with $\sin\xi$ even for values of

* This gives a definite clue regarding the observed *instability* of the outer layers of the chromosphere (see § 16).

ζ only very appreciably different from zero. We can therefore in a *first approximation* neglect sin ξ. We should then have

$$\rho = \rho_0 \left[1 - \epsilon \frac{\alpha}{\beta} \int_0^\zeta \zeta K_1(\zeta) d\zeta \right], \tag{52}$$

where ρ_0 is the density at the base of the chromosphere and ϵ is a constant. Let us further impose the condition that $\rho \to 0$ as $\zeta \to \infty$. This condition would determine ϵ. We have

$$\epsilon = \frac{2}{\pi} \frac{\beta}{\alpha}. \tag{52'}$$

The *density distribution* then is specified by the relation

$$\rho = \rho_0 \cdot \frac{2}{\pi} \int_\zeta^\infty \zeta K_1(\zeta) d\zeta. \tag{53}$$

We notice that in (53) α/β does not appear explicitly. (53) predicts for the march of the *density-gradient* the law

$$\frac{\partial \rho}{\partial \zeta} = -\rho_0 \cdot \frac{2}{\pi} \zeta K_1(\zeta). \tag{54}*$$

In the following table the values of $2\zeta K_1(\zeta)/\pi$ for some values of ζ are tabulated and compared with the purely exponential function which the observations indicate as being very roughly the law realised in the actual chromosphere:—

TABLE II

ζ	$\frac{2}{\pi}\zeta K_1(\zeta)$	$e^{-\zeta}$	ζ	$\frac{2}{\pi}\zeta K_1(\zeta)$	$e^{-\zeta}$
0	0·6366	1·000	2·5	0·1176	0·0821
0·5	0·5273	0·6065	3·0	0·0767	0·0498
1·0	0·3832	0·3679	3·5	0·0496	0·0302
1·5	0·2649	0·2231	4·0	0·0318	0·0183
2·0	0·1781	0·1353	4·5	0·0203	0·0111

Considering the fact that the law (54) is only a first approximation, and considering also the very provisional nature of the observational material, the agreement is very satisfactory (*cf.* fig. 2). The *total mass M* supported in a unit column of the chromosphere is, according to (53),

$$M = \frac{\lambda}{2\pi} \int_0^\infty \rho d\zeta = \frac{\lambda}{\pi^2} \rho_0 \int_0^\infty d\zeta \int_\zeta^\infty x K_1(x) dx$$

$$= \frac{\lambda \rho_0}{\pi^2} \int_0^\infty \zeta^2 K_1(\zeta) d\zeta. \tag{55}$$

* It is easy to convince oneself by quite general arguments that *we should always have the law* (54) *as a first approximation whenever there exists a proportionality of* ξ *to* $\zeta K_1(\zeta)$. The law (54) is therefore quite general, and is not merely the feature of the two-dimensional case we are treating.

Now there is a general formula for the type of integrals which occur in (55) :

$$\int_0^\infty K_\nu(\zeta)\zeta^{\mu-1}d\zeta = 2^{\mu-2}\Gamma\left(\frac{\mu-\nu}{2}\right)\Gamma\left(\frac{\mu+\nu}{2}\right). \tag{56}*$$

We have therefore

$$M = \frac{2\lambda\rho_0}{\pi^2}, \tag{57}$$

or if N is *the number of atoms* † *in the unit column*, and n_0 the number per unit volume at the base, then

$$N = n_0 \cdot \frac{2\lambda}{\pi^2} = 0{\cdot}2026 n_0\lambda. \tag{58}$$

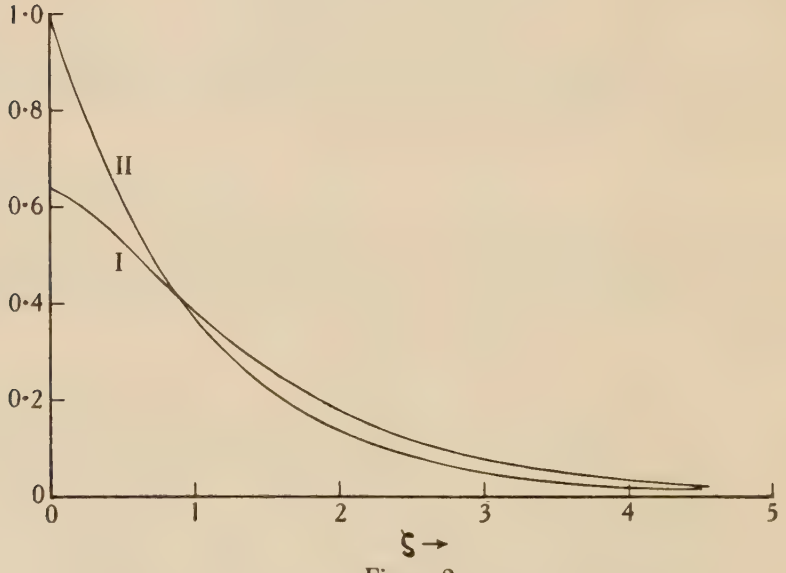

Figure 2.

Comparison of the Predicted March of the Density-Gradient (I) with the Exponential Law (II)

(58) states that *the number of atoms in a unit column of the chromosphere is equal to the number of atoms in a unit column of about a fifth of the wavelength λ filled uniformly at the concentration occurring at the base of the chromosphere.* This is a rather unexpected result and removes practically the whole of the difficulty associated with the hydrogen chromosphere.‡ In any case it suggests a valuable hint towards a solution.

Another quantity which is of interest in this connection is the *mean horizontal velocity* of the chromospheric atoms. Denoting the mean value of ξ by $\bar{\xi}$ we have

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* See G. N. Watson, *Bessel Functions*, p. 388.

† *In that stage of ionization and excitation in which it is directly supported by radiation-pressure.*

‡ *Cf.* Menzel (1), McCrea (2).

$$\bar{\xi} = \int_0^\infty \xi\rho(\zeta)d\zeta \Big/ \int_0^\infty \rho(\zeta)d\zeta = \frac{\pi^2}{16}a, \tag{59}$$

$$= 3{\cdot}901\sqrt{a}. \tag{59'}$$

By (44) this corresponds to about 66·3 km./sec.$^{-1}$, again consistent with the demands of observation.* Actually the velocities obtained appear to be about twice as large as those observed. To get a better fit λ should be nearer 10,000 km. It appears that the scale of length $(\lambda/2\pi)$ provided by values of λ of the order of 8000 (to 6000) km. give on the whole a better agreement with the details of observation in addition to effecting a closer relation between the chromosphere and the "*granulations.*" † I hope to deal with these questions in greater detail in a separate paper.

§ 15. In § 14, in discussing the density distribution and the structure of the chromosphere, we neglected the term in $\sin\xi$ in (51). We will now retain it and examine the consequent modifications in the structure of the chromosphere.

Let us assume that at the base of the chromosphere the variation of density is given by

$$\rho = \rho_0(1 - \epsilon \sin\xi), \tag{60}$$

where ϵ is a quantity unspecified for the present. (51) combined with (60) yields for the density distribution the following law :—

$$\rho = \rho_0\left\{(1 - \epsilon\sin\xi) - \epsilon\frac{\alpha}{\beta}\int_0^\zeta \zeta K_1(\zeta)d\zeta\right\}. \tag{61}$$

Now for a physically admissible density distribution we may expect the density ρ to vanish for some value of ζ finite or infinite. This imposes in the first instance a restriction on the possible values for ϵ. By (61) we should have

$$1 + \epsilon \leqslant \epsilon\frac{\alpha}{\beta}\int_0^\infty \zeta K_1(\zeta)d\zeta = \epsilon\frac{\pi}{2}\frac{\alpha}{\beta},$$

or

$$\epsilon\left(\frac{\pi}{2}\frac{\alpha}{\beta} - 1\right) \geqslant 1. \tag{62}$$

In § 14 we neglected the term in $\sin\xi$ and determined the value for ϵ (given by (52')) from the condition that $\rho \to 0$ as $\zeta \to \infty$. We need now a similar condition to determine ϵ. We argue as follows :—

The atmosphere with a law of density distribution given by (61) has a *sharp boundary*. The equation of the boundary ζ_B is specified by (*cf.* equation (35))

$$Q(\zeta_B, 0) = \frac{\beta}{\epsilon\alpha} - \frac{\beta}{\alpha}\sin\xi. \tag{63}$$

---

* See A. Unsöld, *Ap. J.*, **69**, 209, 1929.

† Smaller values of λ would require larger values for α/β to obtain equal extensions for the chromosphere, and would thus provide large *mean* velocities (*cf.* equation (59)) in spite of the "unit" of velocity V being smaller.

Consider now the case when the relation (62) is an *equality*. Then equation (63) for this value of ϵ reduces to

$$Q(\infty, \zeta_B) = \frac{\beta}{\alpha}(1 + \sin \xi). \tag{64}$$

Now let ζ_0 be such that (*cf.* equation (41))

$$\frac{\alpha}{\beta} Q(\infty, \zeta_0) = 1. \tag{65}$$

Equation (64) then takes the form

$$\sin \xi = \frac{\alpha}{\beta} Q(\zeta_0, \zeta_B), \tag{66}$$

which is the equation of the trajectory of an atom starting at the point ζ_0 (defined in (65)) and which goes off to infinity but with a final zero limiting velocity. An atom starting at a point on the ζ-axis higher than ζ_0 would go off to infinity but with a finite limiting (vertical) velocity. Hence equations (66) and (65) specify the "*natural boundary*" of a chromosphere for a specified value of α/β. *We now assert that the natural boundary must coincide with the actual boundary.* We should then have

$$\epsilon\left(\frac{\pi}{2}\frac{\alpha}{\beta} - 1\right) = 1. \tag{67}*$$

From (61) and (67) we now obtain

$$\frac{\partial \rho}{\partial \zeta} = -\rho_0 \cdot \frac{2}{\pi}\left(1 - \frac{2}{\pi}\frac{\beta}{\alpha}\right)^{-1} \cdot \zeta K_1(\zeta). \tag{68}$$

For $\alpha/\beta = 40$, $2\beta/\pi\alpha = 0{\cdot}0159$, which can be neglected in comparison with unity. To this order of accuracy then (68) and (54) predict the same law for the march of the density gradient. It is also easily verified that formulæ (55), (57), (58) and (59) all continue to be true to this order of accuracy, *i.e.* all correct to within 2 per cent.

In a *normal* chromosphere (with a specified α/β) it is probable that ϵ will be *slightly greater* than that given by (67), in which case the actual boundary defined by equation (63) will be *lower* than the natural boundary. Conversely, if we are given ϵ governing the actual variation of density at the base, then (67) will determine not only the natural boundary of the chromosphere but also the magnitude of the kinetic velocities that will be encountered among the chromosphere atoms.

An examination of the direct eclipse photographs of the chromosphere does indicate that it is in fact characterised by a sharp boundary, though I have not been able to find any explicit mention of this in the literature. Further, the boundary of the chromosphere has a wavy appearance (see

* Conversely if we had started with the "boundary condition" that the natural boundary of the chromosphere must coincide with the actual boundary, then we should have *derived* the law (60) for the variation of density at the base of the chromosphere with ϵ given by (67).

plate ix in Menzel's memoir), and a measurement made on this plate indicates that the actual distance between consecutive "crests" is of the order of 10,000 kilometres—a valuable confirmation of the general ideas on which we are working. Actually if we accept the theory we can from the form of the chromosphere and the known kinetic velocities of the chromospheric atoms (as deduced from the Doppler effect) determine the minute variations in density occurring at the base of the chromosphere.

§ 16. The accurate treatment for the derivation of the density distribution given in the previous section shows that the first approximation of § 14 is in fact a very good one. Also, as has been pointed out, the density law (53) is very much more general than the simplified idealisation on which it has been derived would lead us to expect (see footnote on p. 27). We therefore provisionally accept the law (53) as being valid for a "curved atmosphere" as well, and examine the consequences a step further.

Now, a quantity which is directly related to the actual observations that are made during eclipses is the number of atoms per cm.^{2} column of the chromosphere viewed *edgewise*. (See Menzel (1), §§ 10, 16, especially pp. 242, 243, 266. For a clear understanding of what follows an explicit reference to these parts of Menzel's memoir will be found useful.) We have to compute, then, the number of atoms N_{ζ_1} in the column $-\infty < \eta < +\infty$ located at ζ_1.* Let R be the radius of the star in the *same scale* as ζ measures the vertical. Then

$$N_{\zeta_1} = \int_{-\infty}^{\infty} d\eta \int_{\frac{\eta^2}{2R}+\zeta_1}^{\infty} \frac{2}{\pi} n_0 \,.\, \zeta K_1(\zeta) d\zeta, \tag{69}$$

where n_0 has the same meaning as in equation (58). After some reduction (69) becomes

$$N_{\zeta_1} = \frac{2^{3/2}}{\pi} R^{\frac{1}{2}} \,.\, n_0 \int_{\zeta_1}^{\infty} \sqrt{\zeta - \zeta_1} \,.\, \zeta K_1(\zeta) d\zeta. \tag{70}$$

It is easily verified that the above equation is equivalent to equation (16.15) of Menzel (1) and hence shows that N_{ζ_1} is proportional to the *intensity gradient* along the chromospheric lines.

When $\zeta_1 = 0$ the integral in (70) can be evaluated (see equation (56)). We have

$$N_0 = \frac{32}{3\pi} [\Gamma(7/4)]^2 R^{\frac{1}{2}} \,.\, n_0 = 2{\cdot}868 R^{\frac{1}{2}} \,.\, n_0. \tag{71}$$

If the density law had been

$$n(\zeta) = n_0 \,.\, e^{-\zeta}, \tag{72}$$

we should have had instead of (70),

$$N_{\zeta_1} = 2{\cdot}507 R^{\frac{1}{2}} \,.\, n_0 e^{-\zeta_1}. \tag{73}$$

When $\zeta_1 = 0$, (73) and (70) give practically the same result.

* η is the y-axis measured in the same scale as ζ measures the vertical.

From (70) we have on differentiating

$$-\frac{d\log N_{\zeta_1}}{d\zeta_1}=\frac{\int_{\zeta_1}^{\infty}\sqrt{\zeta-\zeta_1}\,.\,\zeta K_0(\zeta)d\zeta}{\int_{\zeta_1}^{\infty}\sqrt{\zeta_1-\zeta_1}\,.\,\zeta K_1(\zeta)d\zeta}, \tag{74}$$

while for the law (72) we have simply unity on the right-hand side. Though the formula (74) does not give unity for all ζ, it does tend to unity as $\zeta_1\to\infty$, and the greatest deviation from unity occurs at $\zeta_1=0$, where it takes the value

$$-\frac{d\log N_0}{d\zeta_1}=\frac{3}{4}\left\{\frac{\Gamma(5/4)}{\Gamma(7/4)}\right\}^2=0\cdot7296. \tag{75}$$

Hence

$$0\cdot7296\leqslant-\frac{d\log N_{\zeta_1}}{d\zeta_1}<1. \tag{76}$$

Thus the density law (53) cannot seriously be discordant with observations since an exponential law is found roughly to fit them.

§ 17. *Relation of the Chromosphere to the Prominences.*—It was first suggested by Milne and subsequently emphasised by Menzel that the difference between the chromosphere and the prominences is merely one of degree, and that the two phenomena apparently merge into one another directly. Indeed it is precisely from this observation that Menzel deduces the *necessity* for "turbulence," though it has to be mentioned that "turbulence" in the form he advocates will hardly explain this. To secure a close relation between the chromosphere and the prominences we need essentially a hydrodynamical treatment.

The prominences which are definitely associated with the chromosphere are the *Tornado* and the *quiescent* prominences.*

The Tornado prominences are quite small objects, and according to Pettit they generally start off from the outer layers of the chromosphere. The similarity of the appearance of these Tornados with the trajectories for which the inequality (41) is not satisfied (compare plate i in Pettit (*loc. cit.*) and the two trajectories in fig. 1 which start at $\zeta=3\cdot0$ and $4\cdot0$ with $\alpha/\beta=5$ and 10 respectively) suggests at once a possible origin for these. *Firstly*, that they should start off from the topmost levels is merely a consequence of the fact that in the lower levels α/β has some prescribed (mean) value, and that the atoms starting higher up do not satisfy the inequality (41) and in consequence go off to infinity. *Secondly*, we see that if locally the value of λ (or r) *decreased*, then a trajectory originally belonging to a *complete* periodic solution will now belong to the class of trajectories which go to infinity, and this may be the probable origin of the Tornados. If $\sqrt{r\lambda}$ is roughly the same for all the different chromospheric elements, then there would be *no appreciable "sorting out" effect.* This fact has to be especially emphasised,

* We shall use the nomenclature due to E. Pettit. For a summary see his report in the *Publications of the Astronomical Society of the Pacific*, **45**, 15, 1933.

because it has been frequently asserted that a separation of the different elements in the prominences is a necessary consequence of a theory based on the support by radiation-pressure; but this is not always true. It may be added that one may expect velocities of about 10 km./sec.$^{-1}$ for the atoms in the Tornados.

The second type of prominences which are closely related to the chromosphere are the quiescent prominences. These may merely be a temporary lifting up of the *normal boundary* of the chromosphere due to *variations* in *density* occurring at the base. Normally, as we have seen, ϵ in equation (61) will be slightly greater than the value given by (67). If for some reason the value of ϵ slightly *decreased* and became closer to the value given by (67), then the normal boundary of the chromosphere (which is sensitive to small variations of ϵ, near the value (67)) will get enormously lifted up and give rise to a quiescent prominence. That quiescent prominences exhibit considerable "internal motion of a turbulent kind" is nothing else than the exhibition in some relief of the normal conditions obtaining in the chromosphere itself. There should be no more separating out of the different elements in these quiescent prominences than what occurs in the normal chromosphere.

It is of interest also to notice the general similarity of the "fountains" associated with the spot-type prominences and our periodic trajectories (*cf.* fig. 1 with the "fountains" in plate i of Pettit (*loc. cit.*)). Though these "arched prominences" are not especially related to the chromosphere, yet since near the sunspots the conditions obtaining at the base of the normal chromosphere may be realised in an extreme form, the arched prominences may in fact correspond on a much larger scale to the periodic trajectories which on our ideas should be a normal feature in the chromosphere.

Finally, it may be mentioned that the "spikes" which are of such frequent occurrence in a normal chromosphere are likely to correspond to the emission of the moderately high-speed particles, which possibility has already been pointed out in § 12. The limiting velocities there obtained are of the same order as the velocities actually noticed in these "spikes."

Concluding Remarks.—We have not gone into any great length in a full analysis of the observational material, and it would in fact be hardly worth while at this stage of development in the theory; but on the basis of the evidence presented here it should be clear that the study of the monochromatic radiative equilibrium over a non-uniformly radiating surface may well give valuable hints regarding the general structure of the chromosphere and the prominences.

APPENDIX

As we have seen, the functions $\zeta K_{0,1}(\zeta)$ play an important rôle in the theory, and these functions make their appearance not only in the two-dimensional problem here treated, but also in the equations of motion associated with a radiating surface characterised by an intensity function of quite general types of periodicity. To facilitate the analysis of the more

TABLE III

ζ	$\zeta K_0(\zeta)$	$\zeta K_1(\zeta)$	$\int_0^\zeta \zeta K_1(\zeta)d\zeta$
0	0·00 000	1·00 000	0·00 000
0·1	·24 271	·98 538	·09 946
0·2	·35 054	·95 520	·19 658
0·3	·41 174	·91 680	·29 023
0·4	·44 581	·87 374	·37 978
0·5	·46 221	·82 823	·46 489
0·6	·46 651	·78 170	·54 539
0·7	·46 236	·73 520	·62 124
0·8	·45 228	·68 942	·69 246
0·9	·43 806	·64 488	·75 916
1·0	·42 102	·60 191	·82 149
1·1	·40 216	·56 073	·87 960
1·2	·38 221	·52 151	·93 370
1·3	·36 172	·48 431	·98 397
1·4	·34 112	·44 917	1·03 063
1·5	·32 071	·41 608	1·07 383
1·6	·30 073	·38 501	1·11 391
1·7	·28 135	·35 592	1·15 094
1·8	·26 268	·32 872	1·18 516
1·9	·24 481	·30 335	1·21 675
2·0	·22 779	·27 973	1·24 589
2·1	·21 165	·25 777	1·27 275
2·2	·19 639	·23 737	1·29 750
2·3	·18 202	·21 846	1·32 028
2·4	·16 852	·20 094	1·34 123
2·5	·15 587	·18 473	1·36 051
2·6	·14 404	·16 974	1·37 822
2·7	·13 299	·15 589	1·39 449
2·8	·12 270	·14 312	1·40 943
2·9	·11 312	·13 133	1·42 315
3·0	·10 422	·12 047	1·43 573
3·1	·09 596	·11 046	1·44 727
3·2	·08 830	·10 126	1·45 785
3·3	·08 122	·09 279	1·46 755
3·4	·07 466	·08 500	1·47 643
3·5	·06 860	·07 784	1·48 457
3·6	·06 300	·07 126	1·49 202
3·7	·05 783	·06 522	1·49 884
3·8	·05 307	·05 968	1·50 508
3·9	·04 868	·05 460	1·51 079
4·0	·04 464	·04 994	1·51 601
4·1	·04 092	·04 566	1·52 079
4·2	·03 738	·04 175	1·52 516
4·3	·03 435	·03 815	1·52 915
4·4	·03 146	·03 486	1·53 280
4·5	·02 880	·03 185	1·53 613
∞	0	0	1·57 080

general problem it has been thought useful to tabulate here the values of $\zeta K_{0,\,1}(\zeta)$, though the functions $e^{\zeta}K_{0,\,1}(\zeta)$ are available in Watson's *Bessel Functions*.

Another quantity of frequent occurrence is $Q(\zeta, 0) = \int_0^{\zeta} \zeta K_1(\zeta)d\zeta$. In the last column of Table III the values of this function are given. The integral was computed making use of the successive "differences" of the function, but the first two values had to be computed from the series expansion for $\zeta K_1(\zeta)$ because of the singularity in $K_1(\zeta)$ at $\zeta = 0$.

Trinity College, Cambridge :
1933 *October* 19.

THE SPECTRUM OF NOVA RR PICTORIS (1925), 1933.

H. Spencer Jones, F.R.S., Astronomer Royal.

The magnitude of Nova Pictoris, 1925, continues to decrease at a very slow rate. From observations by Messrs. Houghton and Ensor in South Africa, the mean magnitudes for the preceding three years are as follows :—

	m
1930	8·28,
1931	8·41,
1932	8·55,

indicating an approximately uniform decrease per year in the mean magnitude. Superposed on this progressive decrease there are irregular fluctuations, of the order of $\pm$ 0^m·2. The nova is much brighter than its pre-outburst magnitude of 12^m·75, and it seems probable that it will remain permanently brighter.

The slow decrease in brightness of the nova has made it possible again to photograph its spectrum with the objective-prism attached to the Franklin-Adams telescope at the Union Observatory, Johannesburg. Mr. Wood, the Union Astronomer, kindly acceded to my request that an attempt should be made to obtain the spectrum this year, with a view to studying whether any changes had taken place since 1932. In response to this request, two negatives of the spectrum were obtained by Mr. Worssell. The first negative was exposed for 14½ hours during the nights of February 21/22, 22/23 and 23/24 ; the second was exposed for 12 hours during the nights of March 22, 23 and 24.

In forwarding these negatives, Mr. Wood writes : "For the first negative the precautions for shielding the prism and objective from the street lights were the same as used in previous years, but this year they proved insufficient, as you will see, so before the second negative was taken, a light scaffolding was built round the end of the telescope and the whole wrapped in velvet. Mr. Worssell is quite sure that the light which caused the heavy fogging of the first negative could have reached the plate only through the objective during exposure. This negative was developed for 3 minutes in metol-

hydroquinone developer in total darkness, after experiments with other plates from the same batch of sensitising, and the time should have been short enough to have avoided excessive darkening from even an unexpectedly great amount of sky fog.

"The second negative was desensitised in a solution of pheno-safranine and developed for about 2 minutes in rytol (desensitising in pheno-safranine alters the characteristics of hydroquinone) in a faint white light after experiments had been made to make sure that the light was safe for another plate from the same batch. The plate appeared to blacken very rapidly, even when viewed by transmitted light, but most of the blackening disappeared in the fixing bath and this plate would have stood a good deal more developing."

I am much indebted to Mr. Worssell for the great trouble which he took in obtaining the negatives. Both negatives were measured; in addition, contact copies of the February negative were made by Mr. Melotte and new negatives were made from these positives. The negatives thus obtained were free from most of the fogging on the original negative, but showed the bright lines in the spectrum. Two of these copies were measured.

An account of the spectrum in 1931–32 was given in the *M.N.*, **92**, 728–730, 1932. There were no noticeable changes in the spectrum in 1932 as compared with that in 1931. The bright lines appearing in the spectrum in 1933 are substantially the same as were present in both 1931 and 1932, but there are a few marked changes in relative intensities.

The changes as compared with previous years can be summarised as follows :—

(1) The continuous spectrum has become stronger, relatively to the bright lines. It can be traced on the negatives from about λ 3800 to λ 6300, with a maximum of intensity at about λ 4250. The spectra of several B-type stars appear on the negative : these have maxima of intensity at about the same wave-length. Comparing the spectra of various stars on the 1933 negative with the corresponding spectra on the 1932 negative, it is clear that the increase in the intensity of the continuous spectrum of Nova Pictoris since 1932 is real.

(2) The He^+, λ 4686, line has become the brightest line in the spectrum. It may be recalled that in 1932 and 1931 the brightest line was the line of unknown origin at λ 6088, the next most intense lines being $H\alpha$ and λ 4686, which were of equal intensity. In 1928, λ 4686 had been the most intense line, but during that year the λ 6088 line increased considerably in intensity, and in October of 1928 it was not much inferior to λ 4686.

(3) The relative intensities of the two strong unknown lines, λ 5723 and λ 6088, have changed, as they did also in 1928, indicating that these two lines do not have a common origin. The relative intensities of the strongest lines in 1932 and in 1933 are approximately as follows :—

	$H\gamma$	4686	$H\beta$	4944	5723	6088	$H\alpha$
1932	12	40	15	8	30	75	40
1933	9	50	10	5	16	20	25

(4) There appears to be a slight decrease in the Balmer decrement;

relative to $H\beta$ as unit, $H\alpha$ appears somewhat less intense in 1933 than in 1931, whilst $H\gamma$ and $H\delta$ appear somewhat more intense.

The wave-lengths of the lines measured are given in the following table. With the exception of the strongest lines, referred to above, the lines present are for the most part very weak and ill-defined. The wave-lengths are correspondingly uncertain. Some of the lines are merely suspected and may be due to accidental grain aggregates. The mean wave-lengths measured in 1931–32 are given for comparison. The correspondence of a line in the 1933 spectrum with a line in the earlier spectrum gives some evidence of its reality.

The Spectrum of Nova Pictoris, 1933

λ (1933)	λ (1931–2)	Tabular λ	Identification	Remarks
3760·0	3758·9	3759·9	O^{++}	Diffuse : out of focus. Moderately strong.
3835·1	...	3835·4	$H\eta$	Weak and uncertain.
3839·8	...	...	...	...
3887·8	...	3889·1	$H\zeta$	Weak.
3969·2	3969·6	3967·5	Neb.	Broad blend ; ill-defined. Maximum measured.
		70·1	$H\epsilon$	
3988·9	...	...	...	Suspected ; possibly blend.
4039·0	...	...	...	Weak : appears real.
4101·9	4102·0	4101·7	$H\delta$	Broad ; moderately strong.
4112·0	...	...	...	Very weak ; wave-length uncertain.
4122·7	4121·2	...	...	Weak ; appears real.
4138·6	...	...	...	Suspected ; doubtful.
4274·0	...	...	...	Weak ; appears real.
4289·5	4288·2	...	...	Weak ; possibly a blend.
4341·3	4340·3	4340·5	$H\gamma$	Broad, fairly strong.
4367·0	4363·2	4363·1	O^{++}	Very weak ; wave-length uncertain.
4686·9	4687·1	4686·6	He^{+}	Broad, strong.
4701·2	...	...	...	Suspected as wing to preceding.
4861·4	4861·7	4861·3	$H\beta$	Moderately strong ; ill-defined.
4894·8	4899·0	...	...	Real, but weak and diffuse.
4944·7	4943·8	...	...	Stronger than preceding.
4969·1	4968·1	...	...	Broad blend, details uncertain.
4986·1	4989·0	...	...	
5155·3	5152·1	...	...	Centre of broad ill-defined blend.
5173·3	5176·3	?	F^{+}	Weak ; wave-length doubtful.
5275·6	5277·5	?	F^{+}	Real, but ill-defined.
5307·6	5309·3	...	...	Moderate intensity.
5723·4	5722·6	5725·1	F^{++}	Strong.
6087·5	6087·6	...	...	Strong.
6562·8	6562·4	6562·8	$H\alpha$	Strong.

There is no trace of any lines due to neutral helium. The line λ 5875, which would fall where the continuous spectrum is fairly weak, is certainly absent. The line at λ 3760 is clearly shown, though out of focus. Its identification with the *O*III line at λ 3759·9 appears probable, though no trace can be seen of the *O*III lines at λλ 3774·0 and 3791·3.

The spectrum can be described as consisting of a fairly strong continuous spectrum with several broad bright bands superposed. The He^{+} band at

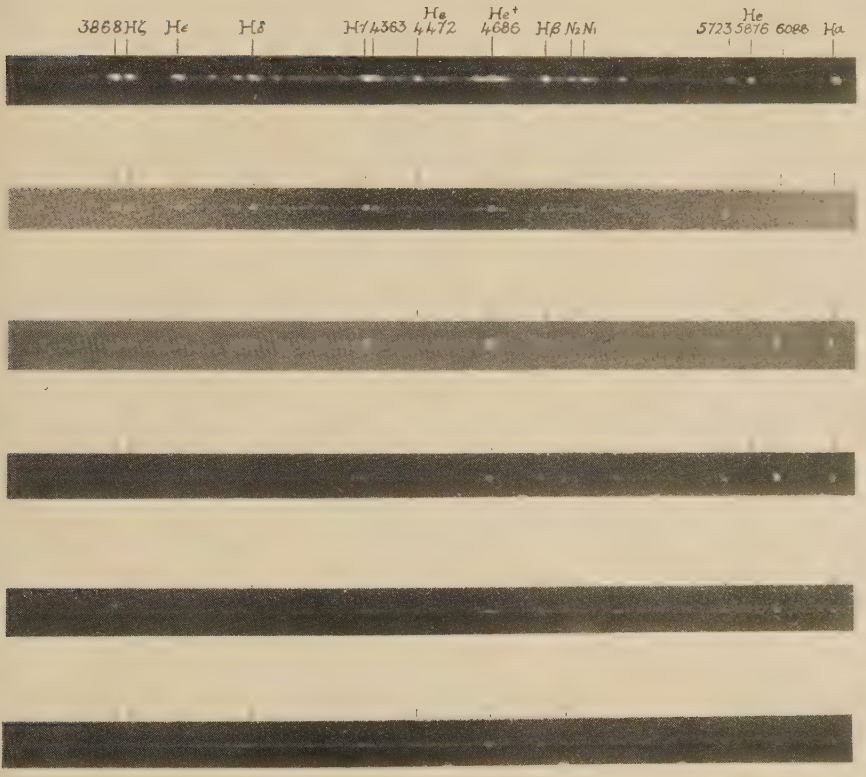
λ 4686 is the strongest of these ; the Balmer series can be detected from $H\alpha$ to $H\eta$; the two bands at $\lambda\lambda$ 6088, 5723 are both strong. The remaining bands are not prominent and, owing in part to blending and in part to their weakness, their wave-lengths are somewhat uncertain. The spectrum has general similarities to spectra of Wolf-Rayet stars, but the complete absence of bands due to neutral helium distinguishes the nova from Wolf-Rayet stars.

Of the two strong lines at $\lambda\lambda$ 5723, 6088, which have formed the most interesting feature of the later spectrum of the novæ, a suggested identification for the line at λ 5723 has recently been put forward by Boyce, Menzel and Payne.* The line is attributed by them to the auroral transition, $^2D^0 - {}^2P^0$, in the spectrum of *F*III, for which the ionisation potential is 60. Some support is given to this identification by the fact that two other lines present in the spectrum of the nova and hitherto regarded as of unknown origin, viz. those at $\lambda\lambda$ 5173, 5275, can be identified as the $^3P_2 - {}^1D_2$ and $^3P_1 - {}^1D_1$ nebular lines of *F*II, for which the ionisation potential is 34·6. These lines appeared during 1926, somewhat earlier than the first appearance of λ 5723. The time of their first appearance cannot be assigned with any precision for they occupy positions near two lines of *Fe*II which were strong in the early spectrum, viz. the line at 5169·0 and the forbidden line at 5273·4. In the discussion of the 1926 spectrum it was noted that the lines at or near these positions were too strong to be attributed entirely to ionised iron.† Of the line near λ 5169 it was remarked, "Stronger on later plates. Mainly Fe^+ on earlier plates, but mainly of unknown origin on later plates." Of the line near λ 5273 it was remarked, "Probably a blend with a line of unknown origin." These two lines have persisted in the subsequent spectrum and, as will be seen from the above table, were still present in the spectrum in 1933. The presence in the spectrum of lines due to *F*II and *F*III would thus seem to be reasonably well established. The strength of the line λ 5723 from 1928 onwards would then indicate a very high abundance of fluorine in the nova. No suggestion has yet been made as to the origin of the other strong unknown line at λ 6088.

It has been suggested by Stratton ‡ that the lines observed in the 1931–32 spectrum at $\lambda\lambda$ 4620·2, 4943·8 and 4989·0 may be the three nebular lines, with normal wave-lengths $\lambda\lambda$ 4635, 4960 and 5006, due to *N*III and *O*III, displaced to the violet by a Doppler shift of 1000 km./sec. I do not think that this identification for these lines can be correct, bearing in mind the general features shown by the earlier spectrum of the nova. During the first 10 months after the outburst, slit spectra with comparison spectra obtained with the spectrograph attached to the Victoria Telescope at the Cape are available. Throughout this period the centres of bright emission bands were very slightly displaced and the bands were practically devoid of structure, in contrast to the bands in Nova Aquilæ, 1918, and other novæ. After March 1926 only objective prism spectra are available, so that absolute

* *Proc. N.A.S.*, **19**, 581, 1933.
† *Cape Annals*, vol. x, pt. 9, 1931 (*vide* p. 137).
‡ *Observatory*, **56**, 26, 1933.

SPECTRA OF NOVA PICTORIS.



H. Spencer Jones, The Spectrum of Nova RR Pictoris (1925), 1933.

displacements of band centres could not be determined. But up to 1929 there is no evidence of Doppler displacements of any band centres relative to the centres of the hydrogen bands. After the disappearance of the multiple absorptions which had been present in some of the earlier stages of the nova, the band widths corresponded to velocities of approach and recession for the violet and red edges respectively of about 430 km./sec. These band widths certainly persisted up to 1929. After 1929 it is not possible to determine the band widths from the spectra available. The spectra afford no evidence of any marked increase in band widths, of noticeable structure in the bands or of displaced isolated maxima. The presence in the spectrum of isolated maxima with displacements of the order of 1000 km./sec. or of bands with centres displaced by this amount seems therefore to be extremely improbable. The ratio of the intensities of the two lines in the spectrum λλ 4944, 4989 is not in accordance with the suggested identification as displaced λλ 4959, 5007 nebular lines, for the line at λ 4944 is appreciably stronger than that at λ 4989, instead of being weaker as would be expected. If the interpretation is correct the line at λ 4944 must be a blend of N_2 with a line of other origin.

Reproductions of the spectra of February and March 1933 are given on Plate 1, together with reproductions of spectra obtained in September 1926, April and October 1928, and February 1931 for purposes of comparison. Of the two 1933 plates, that obtained in February has the better definition but is heavily fogged. The contrast has been increased by copying, the bright lines appearing stronger relative to the continuous spectrum than on the original negative. The reproduction of the March plate gives a better representation of the relative intensities on the original negative. Comparison of the two reproductions serves to show the reality of a number of the fainter bright lines in the spectrum.

The plate shows the main changes in the spectrum during the period from 1926 onwards, which have been referred to in previous papers. Reference may be made to the increase in the Balmer decrement from 1926 to 1928 and from 1929 to 1931, and to its partial decrease from 1931 to 1933; to the increase in strength of the ionised helium line at λ 4686, relative to the other lines, from 1931 to 1933; and to the development of the continuous spectrum between 1931 and 1933.

THE MAGNETIC FIELD OF SUNSPOTS.

T. G. Cowling, M.A., D.Phil.

1. It is well known that a magnetic field is a characteristic property of a sunspot. The essential features of the spot field are as follows:—*

* See Hale, *Ap. J.*, **27**, 315, 1908; *Proc. Nat. Acad. Sci.*, **8**, 168, 1922, and **10**, 53, 1924; and Hale, Ellerman, Nicholson and Joy, *Ap. J.*, **49**, 153, 1919.

(i) At the deepest visible levels the field near the centre of the spot is vertical, and of intensity about 2000 gauss.

(ii) At higher levels the field is much smaller, and the lines of force spread out from the centre of the spot in horizontal directions.*

(iii) When spots occur in pairs, the polarities of the two spots are, in general, opposite. The polarities of the leading spots in one hemisphere are, in general, the same during a sunspot cycle, and opposite from those of the leading spots in the other hemisphere; the polarities of leading spots in either hemisphere are reversed at the next cycle.

If, as seems most likely, the spot field is due to electric currents, these currents must, roughly, flow in a horizontal circuit round the spot: the radial limitation of the field must similarly be due to currents flowing in the opposite sense round a higher circuit. A possible explanation of the existence of such currents was suggested in 1919 by Sir Joseph Larmor,† on the hypothesis that gas flows into the spot column at low levels, and flows out at higher levels. If a horizontal inflow of gas occurs in a vertical magnetic field, an electric force will act on the gas, tending to cause an electric current to flow in a circuit round the axis of the spot, the magnetic effect of the current being to increase the field inside the circuit in which it flows. Larmor suggested that, if the current is sufficiently large, it may be able to maintain the magnetic field to which it is due, just as the current generated in a self-excited dynamo is used to maintain the electromagnets. Similarly, if there is a horizontal outflow at higher levels of the spot column in the presence of a vertical magnetic field, currents producing a radial limitation of the field will flow at such levels. The postulated outflow at surface levels is actually observed,‡ the velocity of outflow being of the order of 1 km./sec.; the existence of an inflow at deeper levels may be inferred.

Larmor's theory is, however, open to some objections. The currents induced in a self-excited dynamo are conducted to a point different from that at which they are set up, so that they can produce a magnetic field at the latter place. No similar conduction occurs in a sunspot, and so the current flowing in a circuit at the base of the spot reinforces the field inside that circuit, but not on it. The analogy with a self-exciting dynamo is, in any case, somewhat misleading, as the current set up in a dynamo is an alternating current, while the current required to maintain the spot field must be unidirectional and fairly steady.

2. Let us therefore consider under what conditions a steady magnetic field can be maintained by the currents it induces in an ionised gas in steady motion in it. Let ρ denote the density of the gas, and $\mathbf{c}$ its (vector) velocity at any point; also let $\mathbf{H}$ be the magnetic intensity. Then, as the state is

* This conclusion is based on the fact that the lines of certain ionised elements, originating at high levels in the spot, show small magnetic fields. The direct observations of the structure of the field are not very conclusive on the point.

† Sir Joseph Larmor, *Brit. Assoc. Reports*, p. 159, 1919.

‡ Evershed, *M.N.*, **69**, 454, 1909; **70**, 217, 1910.

steady, ρ, $\mathbf{c}$ and $\mathbf{H}$ do not depend on the time; also, by the equation of continuity of matter,

$$\text{div } (\rho \mathbf{c}) = 0. \tag{1}$$

In the steady state there will in general be an electrostatic field. The density of electric charge will, however, in general be so small that it is permissible to neglect the part of the electric current arising from the convection of charged matter.* Also in calculating the galvanic current we shall for the present ignore the Hall current, perpendicular both to the magnetic field and the electric force on the gas, and consider only the direct current given by Ohm's law. The electric force on the gas due to its motion in a magnetic field is, in E.M.U., given by $\mathbf{c} \wedge \mathbf{H}$; the electrostatic force is $-\text{grad } V$, where V is the electrostatic potential, which we also suppose measured in E.M.U. Hence, if $\mathbf{j}$ is the electric current-density, and σ the conductivity of the gas,

$$\mathbf{j} = \sigma(\mathbf{c} \wedge \mathbf{H} - \text{grad } V). \tag{2}$$

Also, by ordinary electromagnetic theory,

$$\text{div } \mathbf{H} = 0, \qquad \text{curl } \mathbf{H} = 4\pi \mathbf{j}. \tag{3}$$

It is required to find what solutions of the above equations exist such that $\mathbf{c}$, $\mathbf{H}$ are everywhere finite.

3. The simplest way of attacking this problem is to regard $\mathbf{H}$, ρ, σ as known at all points, and to determine the corresponding values of the other variables. The current-density $\mathbf{j}$ can at once be found from (3). To determine V, take the component of the vector equation (2) in the direction of $\mathbf{H}$. The component of $\mathbf{c} \wedge \mathbf{H}$ in this direction is zero; the component of grad V is dV/ds, where ds is an element of arc of a line of force. Hence, if j_s is the component of $\mathbf{j}$ parallel to $\mathbf{H}$,

$$j_s = -\sigma \frac{dV}{ds}. \tag{4}$$

The general solution of this equation is at once obtained. Let S be a surface intersecting every line of force, and let V be equal to an arbitrary analytic function of position on the surface. Then the value of V at any point P of a line of force which meets S in the point A is given by

$$V_P = V_A - \int_A^P (j_s/\sigma) ds, \tag{5}$$

the integral being taken along the line of force from A to P. As V is a one-valued function, if P is a second point in which the line of force meets S, the previously assigned values of V at P and A must satisfy (5). In particular, if the line of force is closed, then, taking the integral round the whole line of force,

$$\int (j_s/\sigma) ds = 0. \tag{6}$$

* For the case of the Sun's general field, Brunt (*Astr. Nach.*, **196**, 169, 1913) showed that the magnetic field arising from convection of charge is 10^{-15} of that observed. The fraction is not much different for a spot.

To determine $\mathbf{c}$, we set

$$\mathbf{c} = \mathbf{c}' + \alpha\mathbf{H}/\rho, \tag{7}$$

where $\mathbf{c}'$ denotes the component of $\mathbf{c}$ perpendicular to $\mathbf{H}$. Then $\mathbf{c}_\wedge\mathbf{H} = \mathbf{c}'_\wedge\mathbf{H}$, and so equation (2) becomes

$$\mathbf{c}'_\wedge\mathbf{H} = \text{grad } V + \mathbf{j}/\sigma,$$

whence, as $(\mathbf{c}'_\wedge\mathbf{H})_\wedge\mathbf{H} = -H^2\mathbf{c}'$, it follows that

$$H^2\mathbf{c}' = \mathbf{H}_\wedge(\text{grad } V + \mathbf{j}/\sigma), \tag{8}$$

giving $\mathbf{c}'$. The value of the quantity α of equation (7) is found from (1), which becomes, on use of (7),

$$\text{div } (\rho\mathbf{c}' + \alpha\mathbf{H}) = 0;$$

or, as div $\mathbf{H} = 0$,

$$\text{div } (\rho\mathbf{c}') = -\mathbf{H} \cdot \text{grad } \alpha = -H\frac{d\alpha}{ds},$$

ds having the same meaning as before. The solution of this equation runs parallel to that of (4); in the notation of (5), it is

$$\alpha_P = \alpha_A - \int_A^P \frac{1}{H} \text{div } (\rho\mathbf{c}')ds, \tag{9}$$

the value α_A of α at the point A of the surface S being equal to that of an assigned analytic function of position which is such that (9) is true if P also lies on S. Thus, if the line of force is closed, taking the integral round the whole line of force,

$$\int_A^P \frac{1}{H} \text{div } (\rho\mathbf{c}')ds = 0. \tag{10}$$

Hence, for equations (1), (2) and (3) to be soluble, it is only necessary that values can be assigned to V and α on the surface S such that (5) and (9) are satisfied when both P and A lie on S, and that (6) and (10) are satisfied for any closed lines of force. For the value of $\mathbf{c}'$ given by (8) to be finite it is also necessary that

$$\text{grad } V + \mathbf{j}/\sigma = 0 \tag{11}$$

whenever $H = 0$.

4. Observation suggests that, if the magnetic field of a spot is self-contained, then the condition of affairs approximates to the axially symmetric state in which the directions of the magnetic field and of the gas motion at a point lie in the plane through that point and the axis of the spot, while the current-density is perpendicular to that plane. Let us therefore consider the axially symmetric case. Let Oz be taken as the axis of symmetry, and let ϖ denote the distance of any point from this axis, so that $\varpi^2 = x^2 + y^2$. Then V is a function of z and ϖ alone, so that the direction of grad V at any point lies in the plane through the axis and that point. On the other hand, $\mathbf{j}$ and $\mathbf{c}_\wedge\mathbf{H}$ are perpendicular to this plane; hence, by (2), grad $V = 0$, and so we can set $V = 0$.

By (1) there exists a generalised Stokes stream function ϕ, depending on z and ϖ, such that the components of $\mathbf{c}$ parallel to and perpendicular to Oz are

$$\frac{1}{\rho\varpi}\frac{\partial\phi}{\partial\varpi}, \qquad -\frac{1}{\rho\varpi}\frac{\partial\phi}{\partial z}. \tag{12}$$

The components of $\mathbf{H}$ in the same directions can similarly, by (3), be expressed in the forms

$$\frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi}, \qquad -\frac{1}{\varpi}\frac{\partial\psi}{\partial z}, \tag{13}$$

where ψ is a function of ϖ and z, analogous to Stokes's function ; it is such that the total magnetic induction across an area perpendicular to Oz, bounded by a circle with centre on Oz which passes through a given point, is equal to the value of $2\pi\psi$ at that point. Thus ψ vanishes at infinity and on Oz, while, corresponding to some pair of values of ϖ, z, it must have a maximum or a minimum value, for which

$$\frac{\partial\psi}{\partial\varpi}=0, \qquad \frac{\partial\psi}{\partial z}=0, \qquad \frac{\partial^2\psi}{\partial\varpi^2}+\frac{\partial^2\psi}{\partial z^2}\neq 0, \tag{14}$$

the sign of the last expression being positive or negative as the value considered is a minimum or a maximum.

On combining (3) and (13), the magnitude of $\mathbf{j}$ is found to be

$$-\frac{1}{4\pi\varpi}\left(\frac{\partial^2\psi}{\partial\varpi^2}-\frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi}+\frac{\partial^2\psi}{\partial z^2}\right),$$

its direction being as stated above ; the magnitude of $\mathbf{c}_\wedge\mathbf{H}$ similarly is

$$\frac{1}{\rho\varpi^2}\left(\frac{\partial\phi}{\partial\varpi}\frac{\partial\psi}{\partial z}-\frac{\partial\phi}{\partial z}\frac{\partial\psi}{\partial\varpi}\right).$$

Using these values in (2), we obtain the equation

$$\frac{\partial^2\psi}{\partial\varpi^2}+\frac{\partial^2\psi}{\partial z^2}-\frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi}=-\frac{4\pi\sigma}{\rho\varpi}\left(\frac{\partial\phi}{\partial\varpi}\frac{\partial\psi}{\partial z}-\frac{\partial\phi}{\partial z}\frac{\partial\psi}{\partial\varpi}\right).$$

This equation is inconsistent with (14), and so is not satisfied at the point at which ψ is a maximum or a minimum. At this point, in fact, equation (11) is in general not satisfied, and so the value of $\mathbf{c}'$ given by (8) is infinite. A contradiction has thus been reached ; we conclude, therefore, that it is impossible that an axially symmetric field shall be self-maintained.

5. The same argument may be put in a different way, which brings out its physical meaning better. Consider the section of the field by a plane through the axis. In this plane the lines of force are closed curves, the curves $\psi=\text{const.}$, the one inside the other ; these have one or more limiting points, corresponding to maximum or minimum values of ψ. At any such point, by ordinary physical considerations, $H=0$; the same may also be seen from (13) and (14).

Consider a line of force which is a closed curve of infinitesimal dimensions surrounding such a point. Let H_0 denote the mean value of H on the line of force; then the line-integral of the tangential component of **H** taken round it is $H_0 s$, where s is its length. But this is equal to the flux of electric current across the area bounded by the line of force, and, as the mean value of H in this area is less than H_0, this is, by (2), not greater than $\sigma c_0 H_0 S$, where S is the magnitude of the area and c_0 the greatest value of c in it. Thus

$$H_0 s < \sigma c_0 H_0 S.$$

This inequality cannot, however, be valid for finite values of σ and c_0, as S is an infinitesimal of higher order than s. A contradiction has therefore again been reached. We see that, in fact, the currents flowing near the limiting point are too small to maintain the field near that point; hence lines of force will steadily contract and disappear at that point.

A similar argument shows that a field which resembles an axially symmetric field in certain respects cannot, in general, be maintained by the currents it itself sets up. For example, suppose that the lines of force are closed curves all threading a limiting closed curve, and threading it in the same direction. Then the currents required to maintain the field will, near the limit curve, have a component in the direction of the curve which is at all points directed in the same sense round the curve. This component is not due to electrostatic fields, which can only make currents flow from points of high potential to points of low potential, and cannot cause them to flow round a circuit: equally it cannot be due to electromagnetic induction, by an argument similar to the above. Hence such a field cannot be self-maintained.

The same conclusion holds if, instead of being closed curves, the lines of forces are spiral curves threading on the limiting curve, and rotating round it in a fixed sense. In this case H does not necessarily vanish on the limiting curve, which is in general a line of force; but as the electric force acting on matter moving in a magnetic field is perpendicular to the field, electromagnetic induction again cannot produce a current round the limiting circuit. The conclusion is also unaffected when the Hall current is taken into account, as this current is also perpendicular to the magnetic field, and its effect becomes small compared with that of the ordinary induced current as H becomes small.

Since, then, fields possessing a general similarity to an axially symmetric field cannot be self-maintained, we are led to conclude that the magnetic field of a sunspot is not self-maintained. For the same reason the general magnetic fields of the Sun and the Earth cannot be self-maintained, as was suggested by Larmor.

6. While, however, the currents induced by the motion of gas in a sunspot are unable to maintain the magnetic field in the absence of external factors, such currents will produce a large disturbance in any external field that may be present. This suggests the possibility that the sunspot field may arise as a disturbance in the (almost horizontal) general field of the Sun. In such a case, the lines of force will not be closed curves near the spot, but will

enter the spot region from the south, and leave it toward the north ; they enter the spot column near its base or near the surface according as the polarity of the spot is positive or negative.

To show that the observed field can be generated in this way we must show that equations (1), (2) and (3) can be satisfied in the finite region occupied by the spot disturbance, the magnetic field being similar to that of a spot. The analysis of § 3 shows this to be possible. If H vanishes at no point the value of c' given by (8) will always be finite. The electrostatic field must vanish at points outside the spot disturbance ; this is ensured if a relation similar to (6) is satisfied, the line-integral being now along that part of any line of force which lies within the region considered ; it is fairly clear that this relation can be satisfied by a field similar to that of a sunspot.

The electric currents required to generate the spot field will, roughly speaking, flow round the axis of the spot, as was postulated in § 1. The motion which produces these currents will, however, differ somewhat from that postulated there. Since the motion perpendicular to the magnetic field is strongly damped because of electromagnetic induction, the gas tends to move along the lines of force ; on the other hand, when the gas moves across the lines of force, it tends to push these in front of it. In a spot of positive polarity the lines of force enter the spot column from the south at the base, and leave toward the north at the top ; hence the motion must include an uprush of gas in the spot column, the lines of force being held up, as it were, by the moving gas on its southern side, followed by a descent of the gas, on an average at some distance to the north, the directions of the lines of force and of the motion being roughly the same. In a spot of negative polarity, on the other hand, the descending gas will be on the south side of that ascending.

A close correlation will accordingly exist between the polarity of a spot and the type of motion of the gas in it. In the formation of a spot in an undisturbed region the type of motion present may be expected to determine the polarity. The explanation of the connection of polarity with the sunspot cycle may lie along these lines. On the other hand, in the neighbourhood of a spot already existing the vertical component of the magnetic field has the opposite sign from that of the field in the spot ; hence, if a new spot appears in this region, its polarity will tend to be opposite from that of the first, and the motion of gas near it will be modified accordingly.

7. There are three possible quantitative checks on the above hypothesis. First, as the lines of force in a spot are merely those which normally thread the region of the spot disturbance, somewhat displaced in position, the total magnetic flux across the cross-section of the spot column must equal the flux of the general field across the region of the spot disturbance. Now the cross-section which the spot disturbance offers to the general field can hardly be much larger than the cross-section of the spot column ; hence the average intensity of the general field in the regions considered cannot be much smaller than the spot field. In the reversing layer the intensity of the general field does not exceed 50 gauss, but the intensity rapidly increases with the depth, and it is not unlikely that at lower levels the field is con-

siderably larger. For the present hypothesis to be correct, the general field must possess an intensity at lower levels not less than, say, 1000 gauss, a value which, in the absence of data to the contrary, cannot be regarded as impossible.

The second check is that no impossibly large velocity of the gas in a spot shall be required for the currents induced in the moving gas to be able to produce a disturbance in the general field of the magnitude observed. Suppose, for example, that a current flowing round the axis of the spot is responsible for its field. If the radius of the circuit in which it flows is, say, 10,000 km., to produce a field of 2000 gauss at its centre a current of $10^{12}/\pi$ E.M.U. must flow round the circuit; if the cross-section of the circuit is taken as a square of side 1000 km.,* this corresponds to a current-density of about 3×10^{-5} E.M.U. The conductivity at the base of the spot is taken to be 10^{-9} E.M.U., in agreement with the values found in a previous paper.† Thus the electric force producing the above current is 30,000 E.M.U., which is the force acting on matter moving with a velocity of 30 cm./sec. in a field of 1000 gauss. Hence no impossibly large velocities are required to produce a disturbance of the type suggested. It may equally be seen that no impossibly large velocities are required for currents to appear at the top of the spot column large enough to cause the vertical limitation of the field.

The third check is that no impossibly large mechanical effects shall arise in the production of the spot field. In the above a current of density 3×10^{-5} E.M.U. was supposed to flow in the presence of a magnetic field of 1000 gauss; in consequence, a force of magnitude 3×10^{-2} dynes/cm.^{3} acts to retard the motion of the matter across the field. If this force acts at all points inside a circuit whose cross-section is a square of side 1000 km., to overcome it there must be a pressure 3×10^6 dynes/cm.^{2} greater on one face than on that opposite. Hence the pressure at the level of this circuit must be larger than 3×10^6 dynes/cm.^{2}. This is considerably higher than current estimates of the pressure at the base of a spot, and confirmation of this value by other methods is required. If, however, the present hypothesis is valid, the mechanical effect of the currents producing the spot field is extremely important; probably it is one of the chief factors determining the magnitude of the velocities appearing in the spot.

If the motion of gas across the field at the top of the spot column causes currents to be set up which produce the radial limitation of the field, this motion will likewise be retarded. Here pressure cannot maintain the motion, as the pressure required would again have to be of the order of

* The values adopted here for the depth of the base of the spot column, etc., differ greatly from those given by earlier workers on sunspot theory, *e.g.* Milne suggests a depth of, say, 25 km. These estimates must, however, be greatly increased when the effect of an abundance of hydrogen on the molecular weight and specific heat are taken into account, and a value exceeding 1000 km. is probable. The parallel estimates of the pressure at the base of the spot require a similar correction.

† *M.N.*, **93**, 90, 1932. The value in question is a mean of those for σ_1 given in the fourth, fifth and sixth rows of the table on p. 96.

3×10^6 dynes/cm.^{2} at least. It is possible, however, that the currents in question are due to a vertical motion under gravity across horizontal lines of force. If the cross-section of the circuit in which the currents flow is equal in area—though not necessarily in shape—to the circuit in which the spot field is set up, a current-density of the same order as that at the base of the spot will be necessary, and so there will again be a retarding force of 3×10^{-2} dynes/cm.^{3} on the moving gas. For gravity to be able to balance this, the density of the gas must be 10^{-6} grm./cm.^{3}, a value far higher than the commonly accepted figure of 10^{-8} grm./cm.^{3} for the base of the reversing layer. Even if the magnetic field in the region considered is taken to be only 100 gauss instead of 1000, we still obtain the high value of 10^{-7} grm./cm.^{3}. This would appear only to be possible if the currents causing limitation flow at a level lower than that of the reversing layer near the spot ; this presumably implies that the visible levels in a spot are lower than the visible levels of the surrounding photosphere. The observational evidence on this point is conflicting and uncertain.*

8. In conclusion we may indicate what effect the Hall current has, on this theory, on the spot field. The ratio of the Hall conductivity to the ordinary conductivity will be small at the base of the spot, if the above estimate of the pressure at the base is valid ; at the surface, however, the ratio is greater than unity, and even when the effect of electrostatic fields in hindering the flow is taken into account the Hall current will be of the same order as the direct current.

The Hall current arises because the motion of electrons across the lines of magnetic force is retarded more than that of the ions, and so its direction is that of the motion across the lines of force. Hence at the surface it flows first outward, then downward. The consequent magnetic field at higher levels is in a direction round the axis of the spot in a negative sense ; if this is superposed on the field already postulated, lines of force, instead of entering or leaving the spot in directions roughly passing through the axis of the spot, will be spirals round the axis, right- or left-handed as the polarity of the spot is negative or positive. Since the outflow from the spot tends to be along the lines of magnetic force, this suggests that the observed vorticity of a spot should be right-handed for a spot of positive polarity, left-handed for a spot of negative polarity. A correlation of vorticity with polarity is not actually observed ; Hale suggests, however, that this is no evidence that such a correlation does not exist, as the vorticity observed on $H\alpha$ spectroheliograms corresponds to a high level, and may differ from that at a lower level.†

Summary.—The theory proposed by Sir Joseph Larmor, that the magnetic field of a sunspot is maintained by the currents it induces in moving matter, is examined and shown to be faulty ; the same result also applies for the similar theory of the maintenance of the general field of Earth and Sun.

* G. Hale, *The Study of Stellar Evolution*, p. 71, 1908 ; J. and M. A. Evershed, "The Spectrum of Sunspots," *Mem. Kodaikonal Obs.*, 1909. For these references I am indebted to Mr. H. W. Newton. A problem related to this is discussed by Miss Moore in *Ap. J.*, **75**, 298, 1932.

† Hale, Ellerman, Nicholson and Joy, *loc. cit.*

The possibility that the sunspot field may arise as a disturbance in the general field is examined, and it is shown that several of the properties of the spot field are explicable on this hypothesis. Observation must, however, ultimately decide on its validity.

In conclusion, I should like to record my gratitude to Professor Chapman for the interest he has shown in this work and for his advice.

University College, Swansea :
1933 *October* 27.

THE NUCLEI OF TWO PLANETARY NEBULÆ.

G. G. Cillié, M.Sc.

1. *Object of the Investigation.*—Group-parallaxes of planetary nebulæ by van Maanen * and determinations based on galactic rotation by Zanstra † indicate radii for these objects of the order of 5000 astronomical units. Many nebulæ are approximately in equilibrium under the action of gravitation and radiation pressure from the nucleus, combined in some cases with rotation. This indicates that the nuclei are very massive.

Menzel pointed out ‡ that, apart from the above evidence, the less marked galactic concentration of planetary nebulæ compared with other O type stars indicates that they are no farther away than these stars. The relative faintness of their nuclei compared with the O type stars Menzel attributes to their much smaller surface areas, and using Plaskett's dimensions for the O type stars he showed that the nuclei of planetary nebulæ have high mean densities, probably of the order 10^5 gm./c.c.

The probable white dwarf character of these stars was recently demonstrated along similar lines by Zanstra,§ who in addition had at his disposal their effective temperatures as given by his theory of nebular luminosity.

The same suggestion has been put forward by Milne ‖ in his theory of stellar structure. According to this theory a nova outburst is the result of a collapse of a star on to itself, with the consequent setting free of a large amount of gravitational potential energy. This sudden increase in the luminosity causes the atmospheric layers to be blown off, leaving the star to settle down to a much denser configuration. Milne then tentatively suggests that planetary nuclei are old novæ and therefore probably have high mean densities.

The above hypothesis, that the nuclei of planetary nebulæ are white dwarfs, is capable of an observational test. If at the surface of a star the gravitational potential is high, the absorption lines in its spectrum are displaced towards the red according to the general theory of relativity. This is of

* van Maanen, *Mt. Wilson Contributions*, **18**, No. 406, 1930 ; **19**, No. 463, 1933.
† Zanstra, *Zeits. f. Astrophys.*, **2**, 329, 1931.
‡ Menzel, *P.A.S.P.*, **38**, 295, 1926.
§ Zanstra, *M.N.*, **93**, 131, 1932.
‖ Milne, Halley Lecture, 1932.

the nature of a Doppler displacement, and can be observed in practice only provided there exists an independent means of obtaining the radial velocity of the star. In the present case this is provided by the emission spectrum of the nebula. Some slit spectra of the nuclei of N.G.C. 6826 and N.G.C. 2392 obtained by H. H. Plaskett at the Dominion Astrophysical Observatory, Victoria, B.C., between 1923 and 1931, were therefore measured to find whether there is such an Einstein displacement.

Details of the observations are given in Section 2. Descriptions of the plates follow in Section 3, and results of the measurements in Section 4.

2. *Observational Details.*—The following description of the observational conditions under which the plates have been taken has been kindly furnished by H. H. Plaskett :—

"In 1923 a programme for the observation of planetary nuclei was drawn up at Victoria with especial attention to those nuclei with spectra described by Wright from his slitless spectra as 'continuous.' It was hoped that slit

Table I

N.G.C.	α 1900	δ 1900	Nebula			Nucleus	
			m_H	m_T	Int.	m_*	Type
	h m	° ′					
2149	05 48·9	+46 06	13·8	10	0 ; 1·0	13	O7
2392	07 23·3	+21 07	16·9	11	1·4 ; 1·7	11	O7
4593	16 07·0	+12 20	15·5	11	0 ; 1·4	12	O8
6826	19 42·1	+50 17	14·9	9	0 ; —	11	O5

spectra of some of these objects would result in the discovery of stars of spectral class earlier than O5. After some preliminary tests observation was confined to the nuclei of the four nebulæ listed in Table I. The first three columns of this table give the N.G.C. number of the nebulæ and its 1900 co-ordinates. The next three columns give the surface brightness, m_H, of the nebula, its total light, m_T (both in photographic magnitudes and taken from an unpublished discussion), and thirdly, the intensity ratios *He* II(4686)/$H\beta$ and N_2(4959*O* III)/$H\beta$ from Wright's I_q estimated intensities. The larger the intensity ratio the higher the excitation in the nebula ; thus N.G.C. 40 (a very low excitation nebula) gives 0, 0·1 for these two ratios, and N.G.C. 7662 (a very high excitation nebula) 1·2, 3·0. The final two columns of the table give the photographic magnitude of the nucleus (from an unpublished discussion) and the spectral type, estimated from the spectra secured in the course of this work.

"As occasion permitted attempts were made to photograph the spectra of these nuclei with the large spectrograph at the Cassegrain focus of the 72-inch telescope at Victoria. Details of the exposures are given in Table II. The first column contains the N.G.C. number of the nebula and the serial number of the plate, the second column the G.C.T. of the middle

of the exposure, the third column the slit-width in mm. and the fourth the camera used with the spectroscope. In this column the symbol *I* indicates that a spectroscope was used with one prism dispersion, and the other symbols have the following significance: *Moff.*, a camera of 25·4 cm. focal length, aperture ratio *F*/3·3 (four-component lens designed by Moffitt), which gives

Table II

Plate	Date, G.C.T.	Slit	Camera	Dispersion	Emulsion	Exposure	Remarks
II 2149		mm.		mm.$^{-1}$		h	
9468	1923 Sept. 27·96	0·069	I Moff.	81 A.	Grafler 60	2·98	
9492	1923 Oct. 7·98	·066	I Moff.	81 A.	Seed 30	4·00	1 comparison
9619	1923 Nov. 16·84	·066	I Moff.	81 A.	Seed 30	3·67	Plate pre-fogged. Cloudy.
2392							
9548*	1923 Oct. 23·98	·064	I Moff.	81 A.	Seed 30	4·15	
9606*	1923 Nov. 8·00	·066	I Moff.	81 A.	Seed 30	4·60	Plate pre-fogged.
II 4593							
11596	1925 May 6·38	·076	I SS	90 A.	Seed 30	4·05	Plate pre-fogged.
11656*	1925 May 23·35	·076	I SS	90 A.	Eastman 40	5·23	Plate pre-fogged.
6826							
9491*	1923 Oct. 7·72	·066	I Moff.	81 A.	Seed 30	5·00	1 comparison.
19853	1931 Aug. 13·35	·064	I M	29 A.	Eastman 40	5·98	
19893*	1931 Aug. 23·38	·064	I S	49 A.	Barnet Press	5·27	

good definition but a curved field with one prism dispersion; *SS*, a camera of 23·0 cm. focal length, aperture ratio *F*/3 (four-component lens designed by Moffitt), which gives good definition and a flat field over the ordinary photographic region; *S*, a camera of 41·7 cm. focal length, aperture ratio *F*/6 (Hastings triplet), with good definition and a flat field; and finally, *M*, a camera of 71·1 cm. focal length, aperture ratio *F*/9 (Hastings triplet), with excellent definition and a flat field. The fifth column of Table II gives the linear dispersion of the resulting spectra in A.U. per mm. at $H\gamma$, the sixth column gives the type of plate used, the seventh the length of the exposure in hours, and the last column contains remarks. By means of a special arrangement in the dark room it was possible to give the plate a light pre-exposure in order to bring the spectrum on to the flat part of the characteristic curve and hence to increase the contrast for a given exposure time. This is indicated in the last column by the word 'pre-fogged.'

"Generally speaking, the length of the exposure was conditioned by the maximum duration of clear sky which could be obtained in any single night. While most of the spectra have sufficient exposure to enable the spectral class of the nucleus to be estimated, only five plates (marked with an asterisk) have enough density and contrast to be suitable for wave-length determinations. Of these the plate 11656 of N.G.C. II 4593 fails to show sufficiently clearly marked absorption lines on the measuring machine.

"When measuring in these spectra the displacement, $\Delta\lambda$, of the nuclear absorption lines relative to the nebular emission lines, many of the sources of systematic error in radial velocity determinations are avoided. Thus lack of collimation of the spectroscope, non-uniformity of illumination of the collimator lens, and imperfect focus, flexure and failure of temperature control will produce identical displacements of both nuclear and nebular lines. On the other hand, guiding error may introduce, particularly with the wide slit and short dispersion used in this work, spurious displacements, $\Delta\lambda$, of a large amount. Thus with the very short dispersion spectra, such as plates 9491, 9606 and 9548, if the star was guided systematically to one side of the slit, the nebular radiation filling the whole slit width, spurious displacements of the nuclear absorption lines either positive or negative of some 60 km./sec. are possible. With the higher dispersion spectra, such as on plate 19893, the maximum spurious displacement due to guiding error is of the order of 30 km./sec. In the course of a long exposure, however, the nucleus, though difficult to see on the slit, is likely on the average to be guided centrally, especially for N.G.C. 6826, where there is but little atmospheric dispersion and that along the slit ; there is a greater probability of a spurious displacement due to guiding on the atmospherically dispersed image of the nucleus of N.G.C. 2392 which has a lower declination. On the whole, however, the spectra marked with asterisks in Table II should suffice to give a satisfactory measure of the amount of the Einstein displacement, if any, shown by the nuclei of N.G.C. 2392 and 6826."

3. *Description of Spectra and Classification of the Nuclei.*—Before proceeding to detailed descriptions of the spectra of N.G.C. 6826 and N.G.C. 2392, we should like to point out that the absorption lines in both nuclear spectra are faint and diffuse. This is to be expected if these stars are of a white dwarf nature with high surface gravity.* They are, however, by no means as broad as the absorption lines in o_2 Eridani *f* and Oosterhof's star.† Furthermore, comparison between the spectra of these nuclei and that of Y Cygni reveals that the absorption lines in the latter are broader and more diffuse than those in the spectra under discussion. In this connection it is interesting to note that Y Cygni is of type O9·5, mass 15·3 ☉ and mean density 0·158 gm./c.c.‡

(*a*) *N.G.C.* 6826.—The nebular emission spectrum as recorded on both plates corresponds closely to the description given by Wright.§ On plate 19893 the nuclear continuous spectrum extends from below λ 3900 to λ 4950 and is fairly uniform, some of the irregularities being probably due to nuclear emissions. On plate 9491 it is less uniform.

Descriptions of the absorption line spectrum as seen on these plates are given in Table III. The intensities given in columns 2 and 4 were estimated by eye and taking $H\gamma = 10$. The estimates are rendered uncertain in the case of the Balmer lines since they are crossed by bright nebular emissions.

* Anger, *H.C.O. Circular* 352, 1930.
† Öhman, *M.N.*, **92**, 71, 1931.
‡ J. S. Plaskett, *Pub. D.A.O.*, **3**, 304, 1926.
§ Wright, *Pub. L.O.*, **13**, 228, 1918.

In this connection use was made of microphotometer tracings in the case of plate 9491.

TABLE III

Absorption Spectrum of the Nucleus of N.G.C. 6826

Approximate Wavelength	Plate 19893		Plate 9491		Probable Origin
	Int.	Remarks	Int.*	Remarks	
4861·1	> 10	Crossed by nebular emission			$H\beta$
4542·2	5	Some structure	10	Structure	He^+
4340·4	10	Crossed by nebular emission			$H\gamma$
4200·8	4	Broad and shallow	8	Broad	He^+
4101·6	8	Crossed by nebular emission			$H\delta$
3933·1	3	Sharp and narrow			Ca^+

* 4541 taken = 10 as Hγ cannot be estimated.

Using as criterion the ratio of the intensities $\frac{4541}{H\gamma}$, we find that the nucleus of N.G.C. 6826 is of type O5.* In the second column of Table V we give the temperature of this star as determined by different methods. According to Fowler and Milne † the non-appearance of ordinary Helium lines indicates that the temperature of the nucleus of N.G.C. 6826 is at least 30,000° K.

(*b*) *N.G.C.* 2392.—On both plates of this nebula only three nebular emission lines are recorded, N_1, N_2 and λ 4686. On plate 9606 the continuous spectrum reaches from λ 3900 to λ 5050 and on plate 9548 it becomes so faint below λ 4100 that no measurements are attempted below this wavelength.

Descriptions of the nuclear absorption spectra are given in Table IV. For comparison Wright's intensities ‡ are added in column 6. All intensities are reduced to $H\gamma = 10$ as standard.

With the intensity ratios $\frac{4541}{H\gamma}$ and $\frac{4471}{4541}$ as criteria the nucleus of N.G.C. 2392 is of type O7. According to Wright's intensities its type lies between O6 and O8. The presence of lines due to ordinary Helium indicate that the temperature of the nucleus is below 30,000° K.

4. *Results of the Measurements.*—The plates were measured on a Hilger machine kindly lent by Mr. D. A. Jackson. The measurement of the absorption line spectrum of the nucleus was made difficult by the faint and diffuse character of the lines. Measures of the emission lines, on the other hand, could be made with high precision. Each plate was measured in the

* H. H. Plaskett, *Pub. D.A.O.*, **1**, 366, 1922.
† Fowler and Milne, *M.N.*, **83**, 403, 1923.
‡ Wright, *Pub. L.O.*, **13**, 206, 1918.

TABLE IV

Absorption Spectrum of the Nucleus of N.G.C. 2392

Approximate Wavelength	Plate 9606		Plate 9548		Wright's Int.*	Probable Origin
	Int.	Remarks	Int.	Remarks		
4859·0			10	Broad	6	$H\beta$
4713·8			2	Narrow, faint		*He*
4542·2	4	Structure	4	Structure	4	He^+
4472·0			4	Narrow	10	*He*
4387·4					3	*He*
4339·6	10	Well defined	10	Broad	10	$H\gamma$
4201·2	3	Shallow	5		4	He^+
4143·3					3	*He*
4103·0	9		5	Spectrum faint here	12	$H\delta$
4097·6	3				8	N^{++}
4026·0					8	He^+
3969·6	6	Structure			...	$H\epsilon$
3933·8*	6	Broad, structure			...	Ca^+

* Wright, *loc. cit.*, p. 206.

TABLE V

Effective Temperatures of Planetary Nuclei

Method	N.G.C. 6826	N.G.C. 2392
From estimates of line intensities *	30,000°	26,000°
From Wright's intensities *	...	25,000°
By Zanstra's Theory †	26,500°–35,000°	31,000°

* Temperatures for the different O type stars are taken from the paper of Fowler and Milne.‡ Russell's more refined work (Halley Lecture, 1933) gives the same temperatures. On the other hand, it is well known that the temperature scale for the hotter stars is still very uncertain.

direct and reversed direction, and, in addition to the nebular and the nuclear lines, a great number of iron comparison lines were measured along the spectrum.

The results of the measures are contained in Tables VI to IX. The first

† Zanstra, *Pub. D.A.O.*, 4, 200, 1931 ; see also Berman, *L.O.B.*, 15, 430, 1930.

‡ Fowler and Milne, *M.N.*, 83, 403, 1923.

column gives the laboratory wave-length of the line, the second its identification, and in the third column is given the mean wave-length on the plate for measures by the writer. The wave-lengths are calculated directly from the measured positions by means of a Hartmann formula computed from the wave-lengths of the iron comparison lines,* and connected to these wave-lengths by means of a curve of errors.

The wave-lengths in the third column are the means of seven separate measures made at different times over a period of six months. The dispersion in these separate determinations is indicated in the fourth column by the arithmetic mean—without regard to sign—of the residual of each measured wave-length from the final mean. The fifth column contains the similarly computed wave-lengths for two independent measures made by H. H. Plaskett with a similarly computed measure of dispersion in the sixth column.

The weighted mean of the two separately determined wave-lengths is given in the seventh column, the weight being three for C.'s measure and one for P.'s measure. The eighth column contains $\Delta\lambda$ (measured wave-length—laboratory wave-length), and this is reduced to a displacement in terms of km./sec. in the last column of the table. The means for the nebula (sub-table *a*) and for the nucleus (sub-table *b*) are given below each sub-table.

Below each of the tables we give the correction to be applied to our measurements of radial velocity on account of the Earth's orbital velocity.† This enables us to find the radial velocity of the nebula with respect to the Sun. For comparison we add Campbell and Moore's values.‡ We also give the relative displacement in km./sec. of the nuclear absorption lines with respect to the nebular emission lines. In the absence of relative motion between nebula and nucleus or of physical causes which displace the lines, this measures the amount of the gravitational displacement of the absorption lines if such an effect is present.

N.G.C. 6826.—It appears from Tables VI and VII that the absorption lines λ 4541 and λ 4200 in the nucleus of N.G.C. 6826 are displaced towards the red. Taking the weighted mean of the four determinations, giving measures of plate 19893 (high dispersion) weight 2 and those of plate 9491 (low dispersion) weight 1, we find a redward displacement corresponding to a velocity

$$+54{\cdot}3 \pm 7{\cdot}3 \text{ km./sec.}$$

If this displacement is due only to the gravitational field of the nucleus, then, according to the general theory of relativity,

$$M/R = 2{\cdot}4 \times 10^{24} \pm 0{\cdot}32 \times 10^{24},$$

where M is the mass of the nucleus in grams and R its radius in cms. Adopting Zanstra's value of the absolute bolometric magnitude, $0^{m}{\cdot}0$, § and an

* Burns, *L.O.B.*, **8**, 247, 1913.
† Schlesinger, *A.J.*, **10**, 1, 1899.
‡ Campbell and Moore, *Pub. L.O.*, **13**, 77, 1918.
§ Zanstra, *Zeits. f. Astrophys.*, **2**, 329, 1931.

effective temperature 30,000° K, we find that $R = 2{\cdot}1 \times 10^{10}$ cm. or 0·305 of the Sun's radius. Hence we find a mass for it 25·3 ⊙ and a mean density $1{\cdot}2 \times 10^3$ gm./c.c.

TABLE VI

Wave-lengths of Lines in the Spectrum of N.G.C. 6826 from Plate 19893

(*a*) Nebular Emission Lines

Lab. λ	Origin	Measure C.	Dev.	Measure P.	Dev.	Mean	Displacement in A.U.	Displacement in km./sec.
5006·84	N_1	06·69	± 0·10	06·63	± 0·10	5006·68	−0·16	− 9·6
4958·91	N_2	58·87	·09	58·75	·09	4958·84	− ·07	− 4·2
4861·33	$H\beta$	61·07	·03	61·15	·01	4861·09	− ·24	−14·8
4340·47	$H\gamma$	40·46	·05	40·30	·09	4340·42	− ·05	− 3·5
4101·74	$H\delta$	01·62	·03	01·62	·01	4101·62	− ·12	− 8·8
3970·08	$H\epsilon$	70·00	·06	70·22		3970·07	− ·01	− 0·8
3967·51	*Nu*	67·33	·02	67·45	·08	3967·39	− ·12	− 9·1
3868·74	*Nu*	68·67	± 0·02	68·74	± 0·04	3868·69	−0·05	− 3·9
							Mean	−6·8 ± 1·1

(*b*) Nuclear Absorption Lines

Lab. λ	Origin	Measure C.	Dev.	Measure P.	Dev.	Mean	Displacement in A.U.	Displacement in km./sec.
4541·63	He^+	42·33	± 0·06	41·81	± 0·25	4542·20	+0·57	+37·6
4199·86	He^+	00·73	·14	01·00	·03	4200·80	+ ·94	+67·1
3933·66	Ca^+	33·06	± 0·04	33·09	±0·09	3933·07	−0·59	
							Mean	+52·4 ± 10·0

Correction due to Earth's orbital motion at epoch of observation is −1·6 km./sec. Radial velocity of N.G.C. 6826 (referred to the Sun) from the emission lines is −8·4 ± 1·1 km./sec. (Campbell and Moore, −6·3 km./sec.). Relative displacement of lines nucleus-nebula is +59·2 ± 10·1 km./sec.

The large violet displacement of the interstellar K line in Table VI (*b*) is of interest. This displacement corresponds to a velocity of −45·0 km./sec., or, correcting for the Earth's orbital motion, of −46·6 km./sec. Correcting further for the Solar motion,* we find that the *Ca* cloud between us and N.G.C. 6826 is on the whole moving with a velocity of −29·4 km./sec. The galactic co-ordinates of N.G.C. 6826 are $l = 50° 28'$, $b = 12° 8'$.† According to Plaskett and Pearce ‡ such a high velocity of approach in this direction is

* Dufton, *M.N.*, **92**, 688, 1932. † Ohlsson, *Ann. Lund Obs.*, **3**, 1932.

‡ Plaskett and Pearce, *Pub. D.A.O.*, **5**, 167, 1933.

TABLE VII

Wave-lengths of Lines in the Spectrum of N.G.C. 6826 from Plate 9491

(*a*) Nebular Emission Lines

Lab. λ	Origin	Measure C.	Dev.	Measure P.	Dev.	Mean	Displacement in A.U.	Displacement in km./sec.
4861·33	*Hβ*	61·38	± 0·07	61·39	± 0·14	4861·38	+0·05	+ 3·1
4471·48	*He*	71·38	·05	71·37	·09	4471·38	− ·10	− 6·7
4340·47	*Hγ*	40·46	·05	40·50	·02	4340·47	·00	0·0
4101·74	*Hδ*	01·78	·06	01·78	·05	4101·78	+ ·04	+ 2·9
4097·33	N^{++}	97·50	·06	97·34	·02	4097·46	+ ·13	+ 9·5
3970·08	*Hε*	70·26	·05	70·24	·07	3970·26	+ ·18	+13·6
3967·51	*Nu*	67·52	± 0·04	67·41	·01	3967·49	− ·02	− 1·6
							Mean	+3·0 ± 1·7

(*b*) Nuclear Absorption Lines

Lab. λ	Origin	Measure C.	Dev.	Measure P.	Dev.	Mean	Displacement in A.U.	Displacement in km./sec.
4541·63	He^+	42·19	± 0·13	41·52	± 0·22	4542·02	+0·39	+25·8
4199·86	He^+	00·87	·19	00·71	·00	4200·83	+ ·97	+69·3
							Mean	+47·6 ± 14·7

The correction to the radial velocity due to the Earth's orbital velocity at epoch of observation is −2·7 km./sec. Radial velocity of N.G.C. 6826 (referred to the Sun) from the emission lines is −5·7 ± 1·7 km./sec. (Campbell and Moore, −6·3 km./sec.). Relative displacement of lines nucleus-nebula is +44·6 ± 14·8 km./sec.

incompatible with galactic rotation (see their fig. 2). This seems to indicate a large-scale turbulent motion of the interstellar cloud in the direction of N.G.C. 6826.

N.G.C. 2392.—According to Campbell and Moore the nebular lines in this case are double where they cross the nuclear spectrum, indicating a high velocity of expansion for the nebular shell. In case of unequal intensity of the two components this may lead to a displacement of the nebular emission lines on an under-exposed plate. Comparing the radial velocities as given below, Tables VIII and IX, and the value given by Campbell and Moore, we find that this effect is not important on plates 9606 and 9548.

By Tables VIII and IX the absorption lines show considerable variation in displacement from line to line. This may be partly due to accidental errors in the settings on account of unevenness of the grain. Ionised helium

lines are very prominent in the absorption spectrum of the nucleus of N.G.C. 2392. The respective members of the Pickering series are therefore involved in the absorptions at $H\beta$, $H\gamma$, $H\delta$ and $H\epsilon$. On account of uncertainty with regard to the correct wave-length to adopt due to such blending, these lines are disregarded in obtaining the final mean displacement of the absorption lines in Tables VIII (*b*) and IX (*b*). From Tables VIII and IX the relative displacement between the lines nucleus-nebula is $-1{\cdot}3$ and $-15{\cdot}7$, both within the limits of uncertainty in the determination of radial velocity. The evidence then is that the nucleus of N.G.C. 2392 is probably not of a white dwarf character.

TABLE VIII

Wave-lengths of Lines in the Spectrum of N.G.C. 2392 from Plate 9606

(*a*) Nebular Emission Lines

Lab. λ	Origin	Measure C.	Dev. ±	Measure P.	Dev. ±	Mean	Displacement in A.U.	Displacement in km./sec.	Weight
5006·84	N_1	07·72	0·09	08·14	0·49	5007·83	+0·99	+59·3	1
4958·91	N_2	59·44	·12	59·94	·32	4959·57	+ ·66	+39·9	1
4685·81	He^+	86·60	·11	86·70	·09	4866·63	+ ·82	+52·5	1
							Mean	+50·6 ± 3·8	

(*b*) Nuclear Absorption Spectrum

Lab. λ	Origin	Measure C.	Dev. ±	Measure P.	Dev. ±	Mean	Displacement in A.U.	Displacement in km./sec.	Weight
4541·61	He^+	41·19	0·10	42·07	0·09	4541·41	−0·20	−13·2	½
4340·47	$H\gamma$	39·49	·13	39·75	·23	4339·56	− ·91		1
4199·86	He^+	01·09	·28	01·53		4201·19	+1·33	+95·0	1
4101·74	$H\delta$	03·03	·10	02·94		4103·03	+1·27		1
4097·33	N^{++}	97·61	·14			4097·61	+0·28	+20·5	½
3970·08	$H\epsilon$	69·70	·39	69·41		3969·63	− ·45		1
3933·66	Ca^+	33·87	0·52	33·59		3933·80	+0·24		
							Mean	+49·3 ± 22·4	

The correction to the radial velocity due to the Earth's orbital motion at the epoch of observation is +26·5 km./sec. The radial velocity of N.G.C. 2392 (referred to the Sun) from the emission lines is +77·1 ± 3·8 km./sec. (Campbell and Moore, +84·2 km./sec.). Relative displacement of the lines nucleus-nebula is −1·3 ± 22·7 km./sec.

TABLE IX

Wave-lengths of Lines in the Spectrum of N.G.C. 2392 from Plate 9548

(*a*) Nebular Emission Spectrum

Lab. λ	Origin	Measure C.	Dev. ±	Measure P.	Dev. ±	Mean	Displacement in A.U.	Displacement in km./sec.	Weight
5006·84	N_1	08·15	0·14	07·75	0·07	5008·05	+1·21	+72·5	1
4958·91	N_2	60·34	·06	60·23		4960·31	+1·40	+84·7	¼
4685·81	He^+	86·86	·16	86·51	·12	4686·77	+0·96	+61·5	1
								+69·0 ± 3·8	

(*b*) Nuclear Absorption Lines

Lab. λ	Origin	Measure C.	Dev. ±	Measure P.	Dev. ±	Mean	Displacement in A.U.	Displacement in km./sec.	Weight
4861·33	$H\beta$	59·00				4859·00	−2·33		0
4713·14	*He*	13·78	0·10			4713·78	+0·73	+ 40·7	½
4541·61	He^+	42·10	·30	42·52		4542·21	+ ·60	+ 39·6	1
4471·48	*He*	71·99	·07	72·10		4472·02	+ ·54	+ 36·2	1
4340·47	$H\gamma$	38·86	·06	39·45	0·06	4339·01	−1·46		1
4199·86	He^+	01·64	·67			4201·64	+1·78	+127·2	½
4100·74	$H\delta$	02·77	·08			4102·77	+1·03		½
							Mean	+53·3 ± 12·9	

The correction to the radial velocity due to the Earth's orbital motion at the epoch of observation is +29·1 km./sec. Radial velocity of N.G.C. 2392 (referred to the Sun) from the emission lines is +98·1 ± 3·8 km./sec. (Campbell and Moore, +84·2 km./sec.). Relative displacement of the lines nucleus-nebula is −15·7 ± 13·5 km./sec.

5. *Summary.*—Some plates of the spectra of the planetary nebulæ N.G.C. 6826 and N.G.C. 2392, taken by H. H. Plaskett at Victoria, are discussed especially with a view to the suggestion that the nuclei of planetary nebulæ are of a white dwarf nature.

The nuclei are classified and it is found that the nucleus of N.G.C. 6826 is of type O5 with a probable temperature of 30,000° K (according to Fowler and Milne) and that of N.G.C. 2392 of type O7 with probable temperature 26,000° K. These temperatures are of the same order of magnitude as those derived by Zanstra's theory of nebular luminosity.

Measurements of the wave-lengths in the spectrum of N.G.C. 6826 reveal that the nuclear absorption lines are displaced with respect to the nebular emission lines on an average to the extent corresponding to a velocity

+ 54·3 km./sec. Estimates of the dimensions of the nucleus are made on the assumption that this is a purely gravitational effect. It is pointed out that the absorption lines, although of a diffuse character, are much narrower than in the spectra of ordinary white dwarfs and even than in the spectrum of Y Cygni.

In the spectrum of N.G.C. 2392 no evidence of a redward displacement in the nuclear absorption lines is found.

In conclusion, it is a great pleasure to record my thanks to Professor H. H. Plaskett for suggesting the problem, for the loan of the plates, and for making some check measurements, as well as helpful discussions throughout the course of the work ; also to Mr. D. A. Jackson for the loan of his measuring machine, and to Professor E. A. Milne for the loan of the calculating machine.

The University Observatory, Oxford :
1933 *June.*

STELLAR PARALLAXES DETERMINED AT THE ROYAL OBSERVATORY, GREENWICH.

(Communicated by the Astronomer Royal)

The following list is a continuation of the results obtained with the 26-inch telescope, and published annually in the November number of the *Monthly Notices.*

In the column "Proper Motion" the first value given is from meridian observations ; the second is the relative proper motion derived from the photographs. The motions in both R.A. and Dec. are expressed in arc of a great circle.

In calculating the parallax the relative proper motion has been used, *not* the meridian proper motion.

In computing the P.E. an average value of ± 0″·027 has been adopted as the probable error of a single plate of unit weight. This value has been adopted from the mean of 251 determinations.

It is the practice to make two exposures on each plate, the plate being turned through 180° after the first exposure.

The first P.E. given is that of the parallax, the second is the P.E. of the relative proper motion.

Generally six comparison stars of mean magnitude 11–12 have been used, of which good images are obtained with an exposure of 3 minutes. The magnitude of the parallax star has been reduced to equality by means of a rotating sector. For the brighter stars, instead of the method of local desensitisation, a neutral-tinted filter absorbing 4·8 magnitudes is now employed in conjunction with the rotating sector. This filter is permanently mounted on one of the plate carriers, and the carrier is reversed between the exposures to eliminate any distortion due to the filter.

The visual magnitude is taken from the *Bonn Durchmusterung*; the photographic, from the *Greenwich Astrographic Catalogue.*

No.	Name and B.D. No.	Mag. Vis. Photog.	Spectral Type	R.A. Dec. 1900·0	Proper Motion R.A.	Proper Motion Dec.	Parallax	P.E.	No. of Plates
				h m s	″	″	″	″	
1	Br. 48	6·5	F8	0 32 13	−·103	+·081	+·024	± ·007	35
	81° 13	6·8		+81° 56′	−·132			± ·003	
2	O.A. 791	9·4		0 44 56	+·252	+·071	−·006	± ·007	35
	73° 39	9·8		+73° 56′	+·223			± ·003	
3	L. 2651	6·0	Ao	1 23 53	+·076	−·019	−·005	± ·007	29
	65° 175	6·1		+65° 35′	+·056			± ·003	
4	ϵ Cas	3·4	B3	1 47 11	+·038	−·017	+·013	± ·006	36
	62° 320	3·2		+63° 10′	+·036			± ·003	
5	49 Cas	5·0	G5	1 55 58	+·002	−·028	+·018	± ·007	30
	75° 86	6·2		+75° 38′	+·002			± ·003	
6	L.F. 362	7·7	Ko	2 6 22	+·120	−·042	−·013	± ·007	34
	66° 191	8·4		+66° 16′	+·112			± ·003	
7	C. 328	8·5	G5	2 19 36	+·080	−·067	+·005	± ·007	26
	84° 42	8·8		+84° 37′	+·069			± ·004	
8	Br. 402	6·0	Ko	3 8 35	+·081	−·139	+·021	± ·007	31
	84° 59	6·4		+84° 33′	+·044			± ·003	
9	64° 385	8·5	Ko	3 13 13	+·059	−·013	+·007	± ·007	27
		8·7		+64° 46′	+·041			± ·004	
10	Gr. 856	5·5	Ko	4 47 37	+·001	+·025	+·016	± ·007	24
	80° 155	6·3		+81° 2′	+·003			± ·005	
11	P. IV 254	5·5	Aop	4 59 45	+·008	−·018	+·020	± ·007	24
	73° 274	5·5		+73° 49′	+·011			± ·005	
12	A.C. 68° 2150	...		5 13 2	+·011	−·346	+·028	± ·006	27
		11·8		+68° 36′	+·011			± ·005	
13	Gr. 1004	7·0	G5	6 8 3	+·025	−·109	−·005	± ·006	29
	86° 79	7·5		+86° 46′	+·012			± ·004	
14	ρ UMa	5·1	Ma	8 53 32	−·027	+·011	−·004	± ·007	22
	68° 551	6·2		+68° 1′	−·023			± ·005	
15	σ′ UMa	5·2	K5	8 59 37	−·019	−·049	+·010	± ·005	37
	67° 573	6·5		+67° 17′	−·019			± ·003	
16	1 Dra	4·3	K2	9 22 51	−·030	−·031	+·012	± ·007	27
	81° 302	5·7		+81° 46′	−·017			± ·004	
17	30 Cam	5·2	F2	10 18 55	−·058	+·037	+·042	± ·007	29
	83° 297	5·6		+83° 4′	−·061			± ·003	
18	P. X 126	4·7	Ko	10 35 55	+·019	−·015	·000	± ·007	29
	69° 586	6·4		+69° 36′	+·019			± ·003	
19	Gr. 1828	7·0	A2	11 45 56	−·041	−·025	+·012	± ·008	24
	69° 628	7·1		+69° 23′	+·012			± ·004	
20	L. 22609	6·8	A3	11 56 30	−·061	−·033	−·006	± ·009	22
	65° 863	7·1		+65° 30′	−·032			± ·005	

No.	Name and B.D. No.	Mag. Vis. Photog.	Spectral Type	R.A. Dec. 1900·0	Proper Motion R.A.	Proper Motion Dec.	Parallax	P.E.	No. of Plates
				h m s	″	″	″	″	
21	4 Dra	4·7	Ma	12 25 44	−·031	−·049	+·002	±·009	26
	70° 700	6·3		+69° 45′	−·004			±·004	
22	*i* Dra	4·8	Ma	13 48 31	−·008	·000	−·011	±·009	25
	65° 963	6·1		+65° 13′	+·004			±·004	
23	*α* Dra	3·4	Aop	14 1 41	−·052	+·015	+·012	±·009	28
	65° 978	3·8		+64° 51′	−·050			±·004	
24	Gr. 2091	5·3	Ma	14 10 13	−·018	−·042	+·012	±·009	31
	70° 778	6·5		+69° 54′	−·020			±·003	
25	P. XIV 260	4·5	Mb	14 56 0	−·069	+·015	+·017	±·009	26
	66° 878	6·1		+66° 20′	−·057			±·004	
26	L.F. 2843	6·3	Ko	16 43 34	−·035	+·053	+·043	±·010	32
	79° 511	7·4		+79° 6′	−·044			±·005	
27	L. 34738	7·7	Ko	18 29 2	+·031	+·029	+·007	±·009	32
	77° 696	7·7		+77° 30′	+·040			±·004	
28	L. 36039	7·3	Ko	19 3 3	+·021	+·037	+·006	±·009	32
	65° 1319	7·7		+65° 26′	+·016			±·006	
29	75° 690	8·0	F8	19 13 0	+·165	+·052	−·003	±·009	31
		8·4		+76° 1′	+·168			±·006	
30	L. 38306	6·8	Ko	19 54 25	−·012	−·044	+·012	±·008	34
	66° 1256	7·8		+66° 29′	−·003			±·004	
31	C. 3387	7·5	F5	22 1 49	−·124	−·036	+·015	±·009	25
	82° 673	7·5		+82° 23′	−·127			±·007	
32	C. 3389	8·0	F5	22 1 56	−·118	−·037	−·008	±·009	25
	82° 674	8·2		+82° 23′	−·138			±·007	
33	77° 879	7·3	G5	22 54 36	−·122	−·030	−·014	±·008	28
		8·5		+77° 58′	−·121			±·003	
34	77° 879	8·5	G5	22 54 36	−·122	−·025	−·005	±·009	25
		9·2		+77° 58′	−·125			±·003	
35	*π* Cep	4·6	G5	23 4 43	+·023	−·044	+·005	±·007	32
	74° 1006	5·2		+74° 51′	−·012			±·003	
36	Br. 3125	6·2	A2	23 23 3	+·129	+·017	+·019	±·009	29
	69° 1332	5·6		+69° 49′	+·130			±·003	
37	Gr. 4220	6·0	Ko	23 59 30	+·092	+·033	+·026	±·008	28
	66° 1679	6·6		+66° 37′	+·098			±·003	

Nos. 31, 32. C. 3387, 9 Σ 2873.
Nos. 33, 34. 77° 879 Σ 2971.
No. 35. *π* Cep OΣ 489. Brighter component observed.

NOTES ON THE PROBLEM OF SPECTROSCOPIC OBSERVATION OF THE REVERSING LAYER DURING A PARTIAL SOLAR ECLIPSE.

Herbert Dingle, D.Sc.

1. *Introduction.*—In the Imperial College expedition to observe the total eclipse of the Sun of 1932 August 31, of which a report has previously been presented,* certain of the devices adopted seem to be of sufficient general interest to be described in detail, on account of their possible usefulness to future observers. The programme, as stated in the report, was a double one: namely, first to photograph the bright-line spectrum at the cusp of the partially eclipsed Sun up to within a few minutes of totality; and secondly, nearer to totality itself, to photograph the Fraunhofer spectrum near the Sun's limb. These problems can very conveniently be attacked together, since they involve essentially the same apparatus and are concerned with different phases of the partial eclipse. It is therefore a matter of some importance to determine beforehand how the instruments should be set up in order to afford the greatest facility in adapting them successively to the different purposes. It is proposed in this paper to discuss some of the more important features of the arrangements which were made for the eclipse at Montreal (which, it will be remembered, was obscured by clouds). A general account of the equipment has been given in the report, from which figs. 1 and 2 are reproduced for convenience of reference. The points which will be dealt with here are the arrangement of the grating, the manipulation of the image rotator, and the calculation of the position angles of the cusps on the Moon's limb.

2. *The Arrangement of the Grating.*—The general disposition of the apparatus has been described in the report and is clear from fig. 1. We shall consider now the elementary theory of the arrangement, from which the best position for the grating can be determined.

Consider light dispersed by the prisms and incident on the grating at an angle i, which will, of course, be a function of the wave-length, λ. The action of the grating is then expressed by the equation

$$(a+b)(\sin i+\sin\theta)=m\lambda, \tag{1}$$

where $(a+b)$ is the grating space, θ is the angle of diffraction, and m is the order of the spectrum. i and θ are here to be measured in the same direction from the outward normal to the grating. The dispersion after the light leaves the grating is $\frac{d\theta}{d\lambda}$, which, from equation (1), is given by

$$\frac{d\theta}{d\lambda}=\frac{m}{(a+b)\cos\theta}=\frac{\cos i}{\cos\theta}\frac{di}{d\lambda}. \tag{2}$$

* *M.N.*, **93**, 340, 1933.

To this the second passage through the prisms will add further dispersion, which, however, we need not now consider.

There are two possible cases : (i) $\frac{di}{d\lambda}$ positive ; (ii) $\frac{di}{d\lambda}$ negative. ($\frac{di}{d\lambda}$ is

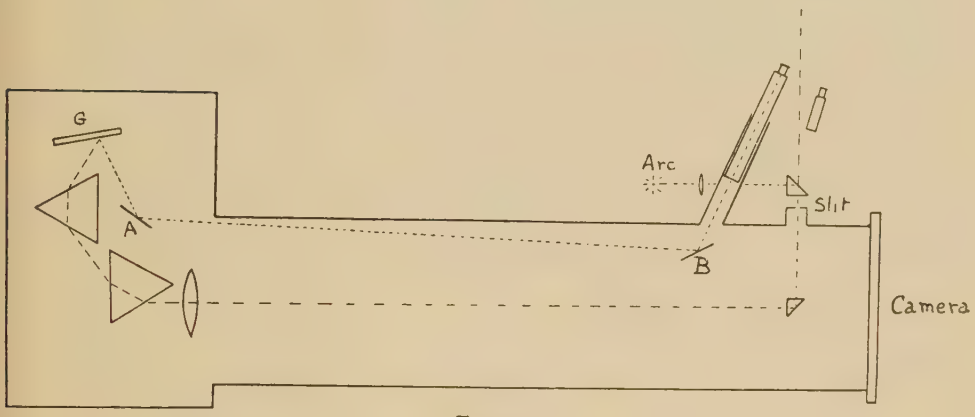


FIG. 1.

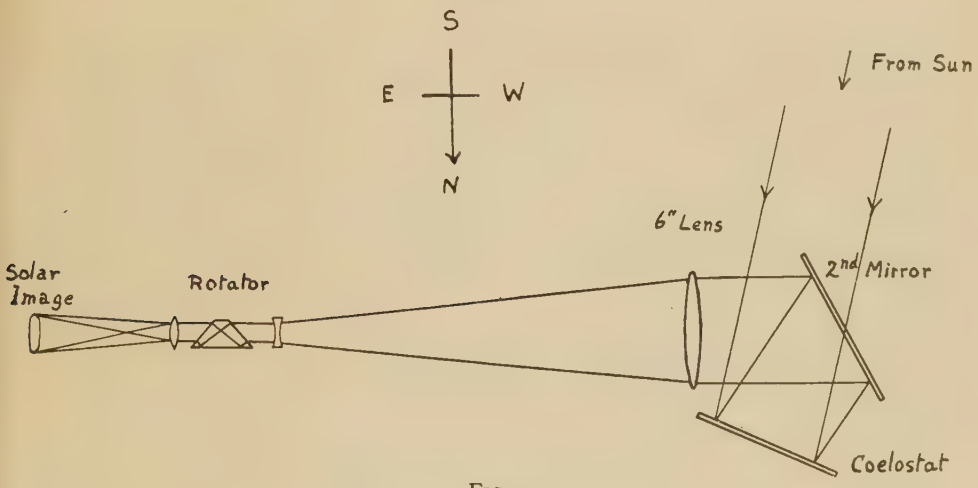


FIG. 2.

clearly the dispersion produced by the prisms before the grating begins to operate.) These cases are distinguished by the orientation of the grating with respect to the light incident on it. Thus, in fig. 3, (*a*), with $i_r > i_v$, represents case (i), and (*b*), with $i_r < i_v$, represents case (ii). Consider the two cases in turn.

Case (*i*).—Referring to equation (2) we see that there are two possibilities, viz. :

$$\frac{\cos i}{\cos\theta}\frac{di}{d\lambda} > \frac{m}{(a+b)\cos\theta} \quad \text{and} \quad \frac{\cos i}{\cos\theta}\frac{di}{d\lambda} < \frac{m}{(a+b)\cos\theta}. \qquad (3)$$

The first possibility gives $\frac{d\theta}{d\lambda}$ negative and numerically less than $\frac{\cos i}{\cos\theta}\frac{di}{d\lambda}$

by $\frac{m}{(a+b)\cos\theta}$, the ordinary expression for grating dispersion with parallel incident light. Since $\frac{\cos i}{\cos\theta} \doteq 1$ for light retracing its path, this means that $\frac{di}{d\lambda}$, the original prism dispersion, is reduced by practically the ordinary

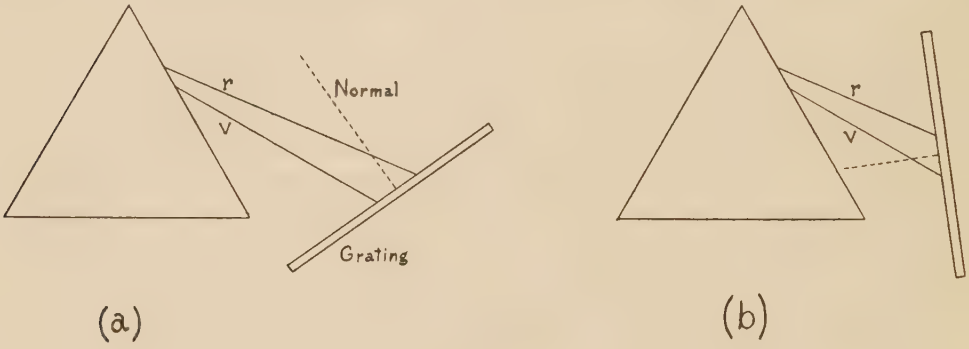


FIG. 3.

grating dispersion. Hence this arrangement is undesirable.

The second possibility gives $\frac{d\theta}{d\lambda}$ positive like $\frac{di}{d\lambda}$, but the resulting dispersion is less than if the grating were used alone, without the prisms. Further, it can easily be seen that the directions in which the rays fall on the prisms on their return journey will be such that the prisms will disperse in the opposite direction to that already accomplished, so that the final effect of the whole system will be to produce a dispersion less than that of one passage through the two prisms alone. Hence this arrangement also is undesirable.

Case (i) is therefore impracticable in all circumstances.

Case (*ii*).—Here we have, numerically,

$$\frac{d\theta}{d\lambda} = \frac{m}{(a+b)\cos\theta} + \frac{\cos i}{\cos\theta}\left|\frac{di}{d\lambda}\right|. \qquad (4)$$

Since $\frac{\cos i}{\cos\theta} \doteq 1$, this means that the dispersion of the light leaving the grating is approximately the sum of the dispersions which would be produced by the prisms and grating separately. Moreover, it is easy to see that the second passage through the prisms adds to the dispersion already produced, so that the net result is a dispersion approximately equal to the sum of the dispersions of four prisms and the grating (in whichever spectrum order it is used).

The arrangement to be adopted is therefore that of Case (ii), illustrated in fig. 3 (*b*). This having been determined, the adjustment of the prisms and grating can be carried out according to principles which are already familiar.

3. *Adjustment of the Image Rotator.*—In order to understand the action of the image rotator, the following points should be noted. Since the aperture will generally be very little greater than is necessary to accommodate the pencil of light (which, as mentioned in the report, must previously be made parallel in order to prevent dispersion in the rotator), it must be constructed so that its length in the direction of travel of the light bears a definite relation to its mean refractive index; otherwise the pencil will be deflected laterally. This can be readily understood from fig. 4, in which (*a*) represents the desired condition and (*b*) the result of providing too great a path in the glass. It is clear also from the figure that even when the rotator is properly constructed, the only rays of the pencil which are transmitted without lateral displacement are those in a horizontal plane whose section is shown by the dotted line in fig. 4 (*a*). We shall call this

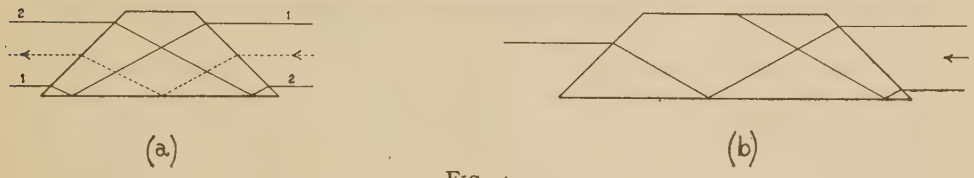


FIG. 4.

line, when the section is made by a vertical plane passing symmetrically through the cylindrical pencil, the *axis* of the pencil. Rays equidistant from the axis on opposite sides of it become interchanged in position, as indicated by the figures 1, 2 in fig. 4 (*a*).

Now the image is turned about the axis of rotation of the *prism*; hence, if it is desired to rotate the image of the Sun about the centre of that image, the axis of the *pencil* must be brought into coincidence with the axis of rotation. In practice, however, it is more convenient to bring about coincidence in the opposite way; namely, first to adjust the direction of the pencil into the line of collimation of the spectrograph, and then to bring the rotating prism into the proper position relatively to it. In the arrangement used at Montreal, the rotator was mounted in a circular cylindrical frame resting at each end on two screws (separated by about 90 degrees) which pressed normally against its surface. The cylinder remained fixed under their action by means of a central controlling spring. The axis of rotation could thus be turned into any direction by suitable manipulation of the screws.

As stated in the report, it is desirable in the cusp problem to rotate the image of the Sun about the centre of the *Moon's* image (which, in the earlier stages of the observations, will be considerably displaced from the centre of the Sun's image), while in the succeeding observations of the Fraunhofer limb spectrum the image of the Sun must be kept stationary. This requires, in the first place, that the cœlostat shall be promptly adjustable to follow either the Sun or the Moon—a requirement which can be met by regulating it for the Sun by observation, and changing when necessary to follow the Moon by calculation of the difference in rates and previous calibration of the rating screw.

The manipulation of the rotating prism is more complicated. Here two essential conditions have to be satisfied : (i) the light falling on the slit must be in the line of collimation of the spectrograph ; (ii) the axis of the rotator must, for the cusp problem, coincide with the axis of the Moon's image. It follows that the light falling on the slit must be that which has passed through the edge of the rotator. The course proposed—and as far as possible found practicable—at Montreal was as follows : With a wide slit, and the cœlostat following the Sun, the light reflected from the centre of the second mirror (see fig. 2) was made to pass along the axis of the spectrograph in the ordinary way, care being taken that the second mirror was illuminated as completely and symmetrically as possible by the cœlostat. The 6-inch lens was then inserted, but since its position had not to be such as to give a sharp image on the slit, a modification of the ordinary process was necessary. The system of the rotator and its two lenses (which, as stated in the report, were mounted on a single base-plate) was placed in the line of light so that the nearer surface of the converging lens was exactly 49·8 cm. from the slit, this distance having been previously determined as the focal length. The 6-inch lens was then placed symmetrically in the line of light and moved to and fro until a sharp image of the Sun was formed on the slit. Since the small converging lens was at its focal distance from the slit, this ensured that it was receiving parallel light from the rotator, and the 6-inch lens was therefore at the correct position. The rotating system was then removed, and the 6-inch lens squared on, by observing light reflected from its surface, and clamped. The rotating system was then again inserted, on a base board containing levelling screws, and adjusted.

The process of adjustment was as follows : The prism rotator having been removed, the two small lenses were placed so that the light from the 6-inch lens passed centrally through them and formed a sharp image of the Sun with its centre on the centre of the slit. The rotator was then inserted and the axis of rotation adjusted by the screws until it rotated the image of the Sun about its centre, the circumference of the image showing no movement while the prism was being rotated. Light from the centre of the Sun was then passing along the axis of the rotator.

With the apparatus thus adjusted, no movement of the rotating system or of the mirrors could prevent the light which fell on the slit from passing along the axis of the spectrograph. That was ensured by fixing the position of the 6-inch lens. It was proposed, therefore, to adopt the following procedure : (i) During the early partial phases of the eclipse the limb of the Sun was to be brought tangential to the slit by first adjusting the line joining the cusps to be parallel to the slit by means of the rotator, and then bringing the Moon's limb tangential to the slit by rotating the second mirror about a vertical axis. With the cœlostat following the Moon, this would give a stationary image of the Moon tangential to the slit. Since, however, the line of collimation was coincident with the axis of the rotator, this image would rotate, not about the centre of the Moon, but about the point of contact of the Moon's image with the slit. Hence (ii) it was proposed to

move the rotating system laterally, guiding rods being fixed for the purpose, until the axis of the rotator was coincident with the axis of the Moon's image, *i.e.* until, on turning the rotating prism, the Moon's image would remain tangential to the slit, since it would rotate about its centre. (This lateral displacement would not, of course, displace the image on the slit; it would only change the axis of rotation.) This having been done, the image would be rotated until the cusp coincided with the slit. As the eclipse progressed, further rotation would then enable the cusp to be followed, with a fixed point of the slit always in tangential contact with the Moon's image.

Transition to the arrangement for photographing the limb spectrum would then be simple. The cœlostat having been adjusted to the Sun's rate, any point of the image of the limb could be brought on the slit by slight movements of the rotator and second mirror.

4. *Calculation of Position Angles.*—Formulæ for calculating the position angles of the cusps on the Sun's limb from the Besselian elements have been given by Comrie.* These are useful in the problem under consideration in order to ensure that the length of exposure is not such as to broaden the spectrum lines excessively on account of the variation of rotation effect with solar latitude. But the position angles on the Sun's limb give no information about the rate at which it will be necessary to rotate the image (which, it must be remembered, moves at twice the rate of the prism rotator) in order to keep the cusp continuously on the slit, for this rotation takes place about the centre of the Moon's image. It is necessary, therefore, to determine the position angles on the Moon's limb, and these, measured from the Moon's north point, are given, according to Comrie, by the expression

$$M + 180° \pm B,$$

where

$$\tan^2 \frac{B}{2} = \frac{(L-m)(L+m-0{\cdot}5459)}{(L+m)(0{\cdot}5459-L+m)},$$

and L, M and m are defined in the paper already referred to.

The rate of movement of the cusps will, of course, vary with the conditions of the eclipse, but it may be of interest to note that in the 1932 eclipse, during a period of ten minutes (which was the maximum length of time which the rotation effect permitted for an exposure) one of the cusps moved over as small a range as $2\frac{1}{2}°$ on the Moon's limb. The problem of following should therefore present no difficulty.

It may conveniently be mentioned here that the "overhang" of the Moon and the extension of the solar atmosphere along the Moon's limb, given respectively by the last formula on p. 484 and the first on p. 485 of Comrie's paper, need more precise definition before the formulæ can be practically applied. I am informed by Dr. Comrie that it would be advisable to substitute for these formulæ the following :—

(i) The "overhang" of the Moon, measured along the solar radius, to

* *M.N.*, **87**, 483, 1927.

the point 1° distant from the cusp as measured along the Moon's limb, is given by either of the expressions

$$\frac{2m \sin 1^\circ \sin A}{2L - 0{\cdot}5459} \times \text{Sun's semi-diameter}$$

or

$$\frac{2m \sin 1^\circ \sin B}{2L - 0{\cdot}5459} \times \text{Moon's semi-diameter.}$$

(ii) The angular distance in degrees, subtended at the Moon's centre, from either cusp to the point where the Moon forms an imaginary cusp with a stratum of the Sun's atmosphere S seconds of arc above the adopted radius, is given by either of the expressions

$$\frac{(2L - 0{\cdot}5459)S}{2m \sin 1^\circ \sin A \times \text{Sun's semi-diameter}}$$

or

$$\frac{(2L - 0{\cdot}5459)S}{2m \sin 1^\circ \sin B \times \text{Moon's semi-diameter}}.$$

In all these expressions the semi-diameters of the Sun and Moon are to be measured in seconds of arc. I understand that these formulæ will be included in the "Explanation" of the *Nautical Almanac* from 1935 onwards.

NOTE ON THE DISTRIBUTION OF THE LUMINOSITIES OF EARLY K STARS.

Gustaf Strömberg, Ph.D.

In the *Bulletin of the Astronomical Institutes of the Netherlands*, **6**, No. 248, 1933, L. Gratton has published his results of a determination of the distribution of the luminosities of the early K-type stars. Since his results are to some extent at variance with those published by me,* a few remarks may be appropriate.

1. The methods used are very similar. Both investigations are based on the solution of a certain integral equation involving the distributions of angular motions, linear tangential motions and absolute magnitudes. The first of these distributions is known from observations, the second can be derived from observations of radial velocities, and the third is to be determined from the first two distributions. Oort † and Gratton have called attention to the fact that the method is not quite new, van Rhijn having used a set of linear algebraic equations in determining the distribution of parallaxes of stars which is equivalent to a solution of an integral equation. Seares has also used an equivalent method.

* *Mount Wilson Contributions*, No. 418 ; *Astrophysical Journal*, **73**, 40, 1931.
† *B.A.N.*, **6**, 239, 1932.

2. Gratton has modified the method by deriving the distribution of linear tangential motion in the τ- and v-directions by assuming that the velocities have an ellipsoidal distribution, the parameters being to some extent dependent upon absolute magnitudes. He has also used the numerical values of τ and v, whereas I used the logarithms of these quantities and also reduced them to zero apparent magnitude. For the τ-components, which are more or less perpendicular to the galaxy, it is obvious that for stars with high galactic concentration a correction for the effect of an ellipsoidal velocity distribution will improve the results. I realized this and gave small and in some cases even zero weight to the results based on the τ-components. As far as the v-components are concerned, I tried to correct for the effect of a non-spherical velocity distribution by comparing the v-component with the radial motions of stars 90° from the proper motion stars. On account of the limited number of stars this method could not be rigorously applied, but there can be little doubt that the effect of preferential motion is nearly eliminated in both my study and in that of Gratton. My numerical method has also the advantage of taking into account the asymmetry in stellar motions in a natural way without introducing any special hypothesis.

The use of logarithms of τ and v seems to me natural, since we are dealing with stellar magnitudes and not with stellar luminosities. The use of logarithms does not in any way give higher weight to small proper motions, as has been suggested, but simply introduces a scale appropriate to the problem, and can have no effect on the results. It was my intention to treat the negative v-components separately; but I found that, when using logarithms, there could be no difference in the results if I combined them with the positive values, although if numbers were used, the natural values would give additional information which is lost if logarithms are used.

3. If we compare the luminosity curves of the stars of spectral type Ko derived by Gratton and by myself, the principal differences are that I have found a greater number of stars of absolute magnitude around $M = +0.3$, a greater number around $M = -3$, and a smaller number between $M = +1$ to $+3$. This is exactly what we would expect on account of the falling off in star density with increasing distance, since then the luminosity curve is dependent on the apparent magnitude range and Gratton in his comparison uses stars of apparent magnitude $m = 5.0$ to 5.8, whereas I combined all stars of apparent magnitude brighter than $m = 6.0$. For this reason it is not quite fair to compare the observed distribution of v-components for stars of $m = 5.0$ to 5.8 with that derived from my distribution of absolute magnitudes of all Ko stars brighter than $m = 6.0$. Nevertheless the agreement is better than might be expected. Further, the fact that I found three fairly distinct groups, denoted by the terms "supergiants," "ordinary giants" and "faint giants," was due to the circumstance that I actually solved the integral equation, whereas Gratton tried a slightly modified error curve which seems to fit the observations sufficiently well. The numerical solution of the equation has a tendency to enhance the effect of small deviations between observed and computed frequencies, and it is at present impossible to tell whether or not there are "dips" on either side of the group of normal K

giants, although the probability of their reality is strengthened by the analysis of the G stars.

4. One of Gratton's conclusions is that there is no gap between giants and dwarfs, and that the gap found by many astronomers is due to a selection in the observing lists. Gratton's curve stops at $M = +4{\cdot}0$, and the inference is that his analysis does not give any stars of types Ko to K2 and with m brighter than 5·8 which have absolute magnitudes fainter than $M = +4{\cdot}0$. But we know of several stars fainter than $M = +4{\cdot}0$ in these intervals of type and apparent magnitude, the following at least: Boss 391, 394, 814, 2573, 4378, 4571, 5009, 5180, 5976, which have absolute magnitudes 6·2, 6·0, 6·2, 5·4, 6·6, 5·8, 6·1, 5·4 and 6·9, respectively. These magnitudes are based on well-determined large parallaxes and can by no possible assumption about systematic errors be made much brighter. *These are the stars which we call dwarfs*, and the deficiency in the absolute magnitude curve lies between $M = +4{\cdot}0$ and $+5{\cdot}0$. Since they are all apparently brighter than $m = 5{\cdot}8$, their existence cannot be due to any selection of large proper motions. Even if a few late type stars having absolute magnitudes between those of giants and dwarfs should be found, our conviction that there is a real deficiency of stars of intermediate luminosity would remain as unshaken as ever.

Carnegie Institution of Washington,
Mount Wilson Observatory:
1933 *September.*

ON THE REFLECTION EFFECT IN ECLIPSING BINARIES.

W. Krat.

(Communicated by Sir Arthur Eddington)

The problem of reflection effect in eclipsing variable stars belongs in its generality to the most complicated problems of stellar radiation. The mathematical calculation of the magnitude of this effect and its variation with phase has been undertaken by J. Stebbins * on the assumption that the effect could be ascribable to diffuse reflection from the photosphere of the secondary. A. S. Eddington,† E. A. Milne ‡ and E. W. Pike § have considered the physical theory of this effect. Professor Eddington argues that the heat-albedo of the secondary must be unity, since its internal temperatures and gradients are too great to be disturbed by the small amount of energy incident from the primary. He has made an important point that the visual efficiency of the re-radiated light might be different from that of the light from the primary. His calculations are limited to the case of widely separated stars. Professor Milne, applying the equations of radiative equilibrium, makes a closely approximate solution of these equations for the case of a parallel beam of radiation incident on a stellar surface. He

* "Probleme der Astronomie" (*Festschrift für H. v. Seeliger*), 442, 1924.
† *M.N.*, **86**, 320, 1926. ‡ *M.N.*, **87**, 43, 1926. § *Ap. J.*, **73**, 205, 1931.

derives the following formula for the variation in the intensity of the re-radiated light with angle :—

$$\mathcal{J}(\alpha, \theta) = \frac{A}{\pi} \frac{\cos \alpha (\cos \alpha + \frac{1}{2})(\cos \theta + \frac{1}{2})}{\cos \alpha + \cos \theta}. \quad (1)$$

Here α is the angle between the incident beam of radiation and the normal, θ is the angle between the normal and the cone of re-radiated energy under consideration and A the energy per unit cross-section incident on the stellar surface. Take a close binary system, as shown in fig. 1. Let O_b and O_f

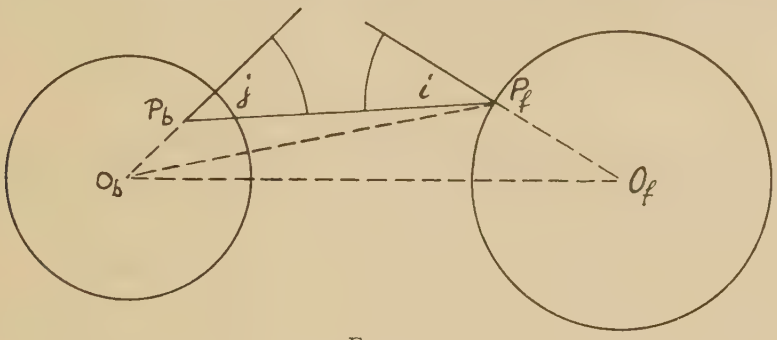


FIG. 1.

be the centres of the stars, P_b and P_f mutually visible points on their surfaces, r_b and r_f the radii of the stars and $L_b = \pi E_b r_b^2$ the relative luminosity of the primary. Consider the reflection at full phase, when as seen from the earth the centre of the secondary is behind the centre of the primary. The energy incident on unit area of the secondary at P_f from an area ds of the primary at P_b is

$$dE_r = \frac{E_b}{\pi} \frac{\cos i \cos j}{d^2} ds \quad (2)$$

where d is the distance $P_b P_f$, i and j are the angles between $P_b P_f$ and the normals at P_b and P_f respectively. Integrating this over the portion of the primary visible from P_f, E. W. Pike * obtained

$$E_r(\theta) = E_b \left(\frac{r_b}{z}\right)^2 \cos \alpha. \quad (3)$$

Here α is the angle between $O_b P_f$ and the normal and

$$z^2 = 1 + r_f^2 - 2 r_f \cos \theta. \quad (4)$$

(The semi-major axis of the orbit is taken as unity.) The formula (3) is valid only within the common inner tangent cone to the two stars. Nevertheless it can be used in further calculations with sufficient approximation since, outside the cone, E_r is small and decreasing rapidly in this region. Then the formula (1) becomes

$$\mathcal{J}(\alpha, \theta) = \frac{L_b}{\pi z^2} \cdot \frac{1}{\pi} \left[\frac{\cos \alpha (\cos \alpha + \frac{1}{2})(\cos \theta + \frac{1}{2})}{\cos \alpha + \cos \theta} \right]. \quad (5)$$

* *Ap. J.*, 73, 205, 1931.

Since at full phase $\cos\alpha=(\cos\theta-r_f)/z$, the law of darkening of the disc of the secondary for reflected light is accordingly :

$$\frac{\mathcal{J}(\theta)}{\mathcal{J}(0)}=\frac{8}{9(1-r_f)}\left[\frac{(\cos\theta-r_f)\left(\cos\theta-r_f+\frac{z}{2}\right)(\cos\theta+\frac{1}{2})}{\cos\theta(1+z)-r_f}\right]. \tag{6}$$

The reflected light thus shows a complete darkening at the limb.

The total light reflected to the earth inside a small solid angle $d\Omega$ from an element of area $d\partial$ of the secondary is

$$\mathcal{J}(\theta)\cos\theta d\Omega d\partial$$

where

$$d\partial=r_f^2\sin\theta d\theta d\psi.$$

The ordinary radiation from the whole disc of the secondary is $L_f d\Omega/\pi$. Hence the ratio s for the whole disc is

$$s=\frac{\text{reflected light}}{\text{ordinary light}}=\frac{2\pi r_f^2}{L_f}\int_0^{\arccos r_f}\mathcal{J}(\theta)\cos\theta\sin\theta d\theta. \tag{7}$$

Let $\bar{z}$ be the mean value of z inside the cone of radiation. Then

$$s=\frac{L_b}{L_f}\frac{r_f^2}{\bar{z}^2}\phi, \tag{8}$$

ϕ being a function of r_f alone (Table I).

TABLE I

r_f	0·0	0·1	0·2	0·3	0·4	0·5
ϕ	...	·66	·59	·52	·44	·35

To find the mean value of z we proceed as follows. Take the intensity of incident radiation ($E_r(\theta)$) in the approximate form derived by Pike :

$$E_r(\theta)=A_bE_b\cos^n\theta, \tag{9}$$

where n is obviously a function of r_f alone, its mean value being near to 3.

Let us consider the disc of the secondary divided in two parts of equal integral intensity (E). Then

$$-nA_bE_b\int_{\frac{\pi}{2}}^{\bar{\theta}}\cos^{n-1}\theta\sin^3\theta d\theta-A_bE_b\cos^n(\bar{\theta})\sin^2(\bar{\theta})=\frac{A_bE_b}{n+2}; \tag{10}$$

hence

$$\cos^{n+2}(\bar{\theta})=0\cdot5 \tag{11}$$

$$(n=3\cos(\bar{\theta})=0\cdot87).$$

Inserting (4), we obtain

$$\bar{z}^2=1+r_f^2-1\cdot74r_f. \tag{12}$$

To compute the variation of the reflection effect with phase we can use Eddington's * formula :

$$L_r(\epsilon) = L_r(\pi)\,\frac{\sin\epsilon - \epsilon\cos\epsilon}{\pi}, \qquad (13)$$

as well as that proposed by Pike as being a better approximation :

$$L_r(\epsilon) = L_r(\pi)\,\cos^{9/2}\left(\frac{\pi-\epsilon}{2}\right), \qquad (14)$$

(ϵ being the phase-angle varying between 0 and π), because it cannot be determined practically with sufficient accuracy.

Published values of the reflection coefficient $\left(\frac{L_r(\pi)}{2}\right)$ have been computed on the assumption that the brightness varies as $1-\cos\epsilon$ instead of (13). As pointed out by Eddington,† this introduces no inaccuracy, because in order to separate the reflection effect from the spheroidal effect the determination is made to depend on the difference of brightness at phases ϵ and $\pi-\epsilon$. For such comparisons the expressions $\frac{1}{2}(1-\cos\epsilon)$ and $1/\pi(\sin\epsilon-\epsilon\cos\epsilon)$ are equivalent.

The computed values of $L_r(\pi)$ represent evidently only the differential reflection effect of the system, because the reflection of the increased light

TABLE II

No.		ρ	r_b	r_f	$L_r(\pi)$ (observ.)	ΔL_r (O − C)	K	*Sp* Prim.	*Sp* Sec.	Authority
1	U CrB	·02	·15	·28	·062	−·010	·0069	B	B3	Baker (pg) [1]
2	β Per	·04	·20	·24	·045	+·004		B8	[F8]	Stebbins (pe) [2]
			·27	·19	·052	+·026	·0019			Danjon (vis) [4]
3	RS Vul	·01	·20	·27	·078	+·033		B8	B9	Dugan (vis) [3]
					·066	+·021	·0038		[F5]	Baker (pg) [6]
4	Y Cam	·02	·24	·24	·041	−·005		A	[F]	Dugan (vis) [3]
5	TX Cas	·02	·52	·27	·030	+·005		B8	[A1]	McDiarmid (vis) [5]
6	RZ Cas	·08	·25	·29	·063	−·005	·0048	A2	[G3]	Dugan (vis) [7]
7	*u* Her	·04	·41	·41	·036	+·006		B3	B3	Baker (pe) [8]
			·29	·29	·030	·000	·00053		[B9]	Baker (pg) [9]

In the last column of Table II the following contractions are used :—

pe = photoelectric, pg = photographic, vis = visual or photometric.

[1] *Laws Obs. Bull.*, No. 29, 1921.
[2] *Ap. J.*, **32**, 185, 1910.
[3] *Contr. Princ. Univ. Obs.*, No. 6, 1924.
[4] *Ann. d. l'Obs. d. Strasbourg*, **11**, 1er f., 1928.
[5] *Contr. Princ. Univ. Obs.*, No. 7, 1924.
[6] *Laws Obs. Bull.*, No. 32, 1921.
[7] *Contr. Princ. Univ. Obs.*, No. 4, 1916.
[8] *Lick Obs. Bull.*, No. 378, 1926.
[9] *Laws Obs. Bull.*, No. 28, 1917.

* *M.N.*, **86**, 320, 1926. † *Ibid.*

of the secondary (L_f) may be also in the same way reflected from the primary. In the case when the components have nearly equal dimensions and luminosities (*e.g.* FS Car) * the reflection effect cannot be determined from the light-curve. Thence the true ellipticity coefficient must be always greater than that computed from the light-curve.

(The intensity of the system as seen from the earth will be at phase-angle $\pi/2$ smaller for $(\pi-2)L_r(\pi)/\pi$ than that at phase-angles 0 and π.)

I have collected in the Tables II and III the results for those variables in which the reflection effect has been closely studied.

TABLE III

No.		ρ	r_b	r_f	$L_r(\pi)$ (observ.)	ΔL_r (O − C)	K	*Sp*		Authority
								Prim.	Sec.	
1	SZ Her	·30	·32	·34	·030	−·038	·005	A3	[F1]	Dugan (vis) [1]
2	R CMa	·19	·24	·24	·015	−·029		F0	[K3]	Dugan (vis) [1]
3	RS CVn	·01	·09	·29	·034	−·019		F4*n*	G8	Sitterly (vis) [2]
4	δ Lib	·06	·28	·30	·043	−·016		A0	[G0]	Stebbins (pe) [3]
5	RY Agr	·06	·19	·24	·000	−·039		A3	[G5]	Dugan (vis) [1]
6	β Lyr	$5\cdot10^{-4}$	·44	·20	·020	−·023		B3*e*	B9	Danjon (vis) [4]
7	Z Dra	·19	·24	·26	·032	−·019	·009	A5	[K1]	Dugan (vis) [5]
8	W UMi	·04	·36	·31	·038	−·031		A0	[F6]	Dugan (vis) [6]
9	U Cep	·02	·19	·31	·043	−·028		A0	[G3]	Dugan (vis) [7]
					·040	−·031				Baker (pg) [8]
10	TV Cas	·05	·27	·30	·074	−·008		B9	[F5]	McDiarmid (vis) [9]
					·032	−·050				Cummings (pg) [10]
11	X Tri	·14	·26	·30	·018	−·068		A3	G5	Dugan (vis) [11]
12	RV Oph		·12	·20	·018	−·010	·23	A0	[G1]	Dugan (vis) [12]
13	Z Vul	·06	·29	·27	·030	−·015		B3	{B3 / [A5]}	Baker (pg) [13]
14	TW Dra	·01	·21	·32	·008	−·057		A5	[G4]	Baker (pg) [14]
15	RT Per		·31	·27	·022	−·022	·0024	F	[G5]	Dugan (vis) [15]

[1] *Contr. Princ. Univ. Obs.*, No. 6, 1924.
[2] *Contr. Princ. Univ. Obs.*, No. 11, 1930.
[3] *Publ. Washburn Obs.*, **15**, 1, 1928.
[4] *Ann. d. l'Obs. d. Strasbourg*, t. ii, 1er F, 1928.
[5] *Contr. Princ. Univ. Obs.*, No. 2, 1912.
[6] *Contr. Princ. Univ. Obs.*, No. 10, 1930.
[7] *Contr. Princ. Univ. Obs.*, No. 5, 1920.
[8] *Laws Obs. Bull.*, No. 30, 1921.
[9] *Contr. Princ. Univ. Obs.*, No. 7, 1924.
[10] *Laws Obs. Bull.*, No. 27, 1917.
[11] *Contr. Princ. Univ. Obs.*, No. 8, 1928.
[12] *Contr. Princ. Univ. Obs.*, No. 4, 1916.
[13] *Laws Obs. Bull.*, No. 26, 1916.
[14] *Laws Obs. Bull.*, No. 33, 1921.
[15] *Contr. Princ. Univ. Obs.*, No. 1, 1911.

We see that in the first group of variables (the colour range primary—secondary being relatively small) the formula (8) may be applied with some confidence. The residuals (O − C) are small, excepting RS Vul, and have mostly the same sign, as pointed out by Eddington.† The factor K of

* *Veröff d. Stern. Ber. Bab.*, **9**, 5, 1932. † *M.N.*, **86**, 320, 1926.

density distribution computed from the advance of periastron by means of Russell's * formula :

$$\frac{\Delta w}{P} = K_b r_b^5 \left(1 + 7\frac{m_f}{m_b}\right) + K_f r_f^5 \left(1 + 7\frac{m_b}{m_f}\right) \qquad (15)$$

(P being the period of revolution of the binary, Δw the arc covered during P by the advancing periastron and m_b, m_f the masses of the components), and the mean densities (ρ) are given in order to form an idea of the nature of the stars.†

The variables of the second group having considerable colour ranges show large negative residuals. These residuals seem to be independent of the densities and the internal constitution of the stars. The photographical values of the reflection effect for all eclipsing binaries studied are always smaller than the visual ones, the photo-electrical values being intermediate.

Till now the relative orbits of both the components were considered to be circular. In the case of elliptical orbits the reflection effect will be a function of the true anomaly.

If the longitude of periastron (w) is near 0 or π, the effect of brightening at periastron due to reflection of light is given by

$$2p_r = \frac{4e(r_f)^2 \phi L_b f(\pi/2)}{\bar{z}^3}. \qquad (16)$$

In more general cases (w deviating largely from 0) the equation (16) may be written thus :

$$2p_r = \frac{L_b \phi r_f^2}{\bar{z}^2}\left[\frac{2e}{\bar{z}}(f(\epsilon_0) + f(\pi - \epsilon_0)) + (f(\epsilon_0) - f(\pi - \epsilon_0))\right] \qquad (17)$$

(the reflection by the primary star is neglected), ϵ_0 being the phase-angle at periastron. Practically it is usual to compute the variation of light between the minima in the form :

$$l = l_0 + a\cos t + b\sin t + c\cos^2 t \qquad (18)$$

(t is the phase-angle varying from 0 to 2π), where b denotes the half amplitude of the periastron effect. This formula is exact for computing the range of variation of the reflection effect only in the case $w \sim 0, \pi$.

The author has calculated the theoretical value of the periastron effect in the case of Boss 46.‡ The light-curve and variation of radial velocities of this eclipsing variable were investigated by Pearce. In this case the coefficient at $\sin t$ was found to be equal $+0{\cdot}025$, w being very near to 0 and $e = 0{\cdot}038$.

Using (16) we have $2p_r = 0{\cdot}011$. We see that the computed values of the periastron effect in the system Boss 46 are smaller than the observed. It seems to be the case of other eclipsing variables, RT Per, W Cru and RS CVn. In the light-curve of Boss 46 there is no evidence of the reflection

* *M.N.*, **88**, 641, 1928. † *V.F.P.A.*, "V.S.," B. iv, No. 7, 1933.
‡ *Publ. D.A.O.* (*Victoria*), **3**, 13, 1926.

effect. In the systems RT Per and RS CVn it is small and relatively insignificant.

It is of interest that the periastron effect was observed only in the systems of small orbital eccentricity (e). Therefore the increasing of reflection effect near periastron represents no explanation of the periastron effect, the origin of which is nearly connected with tidal "friction" and tidal pulsation of the components. The last is not merely a variation of superficial ellipticity of stars due to tidal effects, but something like the cepheid pulsation. (The study of this effect is in progress.)

The orbital elements of the binary (e, w) computed in the usual way * or their limits † will be appreciably in error due to the extensive darkening at the limb of the secondary. (The length of the secondary minimum will be shortened.) The necessary correction in this case can be easily computed by the introduction of an effective radius of the secondary as seen at full phase.

The change in spectral type due to reflection was completely discussed by Milne.‡ The possible effects of reflection on solutions of a light-curve were discussed by Pike,§ and an approximate correction to the spectroscopically determined masses of a binary (to account for the asymmetrical brightening of the secondary as seen at quadrature) was computed.

Kasan, Engelhardt Observatory :
1933 *August.*

A METHOD FOR COMPUTING ORBITS OF COMETS AND MINOR PLANETS WHEN LONG TIME INTERVALS ARE AVAILABLE.

Rev. M. Davidson, B.A., D.Sc.

The present paper describes the computation of the orbit of a comet or minor planet that has been observed over a considerable period ; it is hoped that the method will prove easier for use with a computing machine than some of the methods usually adopted. It is assumed that a preliminary orbit has already been computed, and from this the approximate geocentric distances at the times of the first and third observations are determined. These will generally be accurate to the first decimal place, and very often beyond this ; it is advisable, therefore, to use the full number of figures at the beginning, because in some cases they may not differ much from the true value, though, generally speaking, the agreement does not extend beyond the second decimal place.

Geometrical Equations.—The direction-cosines of the comet and also the equatorial (or ecliptic) rectangular co-ordinates of the Sun enable the well-

* *V.F.P.A.*, "V.S.," B. iv, No. 4, 1933.
† *B.A.N.*, **7**, 241, 1933.
‡ *M.N.*, **87**, 43, 1926.
§ *Ap. J.*, **73**, 205, 1931.

known fundamental equations, expressing the condition that the Sun and the various positions of the comet are in a plane, to be obtained. These equations, which are geometrical and are independent of dynamical considerations, are

$$\begin{aligned} c_1\rho_1 l_1 - \rho_2 l_2 + c_3\rho_3 l_3 &= c_1X_1 - X_2 + c_3X_3 \\ c_1\rho_1 m_1 - \rho_2 m_2 + c_3\rho_3 m_3 &= c_1Y_1 - Y_2 + c_3Y_3 \\ c_1\rho_1 n_1 - \rho_2 n_2 + c_3\rho_3 n_3 &= c_1Z_1 - Z_2 + c_3Z_3 \end{aligned}$$

If the ecliptic is taken as the plane of reference, the right-hand term of the last equation is zero.

The method for equating the geometrical and dynamical conditions will now be described. From approximate values of ρ_1 and ρ_3, the values of c_1 and c_3 are first computed, and from these ρ_2. Then an arbitrary increment is given to ρ_1, ρ_3 remaining the same, and new values of c_1, c_3 and ρ_2 are computed. The process is repeated, ρ_3 in this case receiving the increment and ρ_1 remaining unchanged. New values of c_1, c_3 and ρ_2 are found, and it is then a simple matter to express c_1 and c_3 as functions of $\Delta\rho_1$ and $\Delta\rho_3$. Thus, if $+0{\cdot}01$ is the increment (a very convenient one in nearly all cases), the method can be represented as follows. Let ρ_1' and ρ_3' be the provisional values of the first and third geocentric distances, and let c_1', c_3' and ρ_2' be the values of c_1, c_3 and ρ_2 derived from these. When ρ_1 receives an increment $+0{\cdot}01$, let c_1'', c_3'' and ρ_2'' be the corresponding values, and when ρ_3 receives an increment $+0{\cdot}01$, let these be c_1''', c_3''' and ρ_2'''. Then the values of c_1 and c_3 on geometrical grounds only are

$$\begin{aligned} c_1 &= c_1' + \frac{c_1'' - c_1'}{0{\cdot}01}\cdot\Delta\rho_1 + \frac{c_1''' - c_1'}{0{\cdot}01}\cdot\Delta\rho_3 \\ c_3 &= c_3' + \frac{c_3'' - c_1'}{0{\cdot}01}\cdot\Delta\rho_1 + \frac{c_3''' - c_1'}{0{\cdot}01}\cdot\Delta\rho_3 \end{aligned} \qquad (1)$$

These expressions can also be obtained by differentiation ; reference to this will be made later in the example.

Each set of geocentric distances is used to compute the heliocentric distances and also the angles between the radii vectores from the Sun to the comet. From these the values of c_1 and c_3 are computed from dynamical considerations for the three cases, and again c_1 and c_3 can be expressed as functions of $\Delta\rho_1$ and $\Delta\rho_3$. These are of the same form as (1), remembering that c_1', c_3', $c_1'' - c'$, etc., are now different numerically from those in (1). If C with subscripts, etc., denote the dynamical values, then, since C and c must ultimately be equal,

$$\begin{aligned} c_1' + \frac{c_1'' - c_1'}{0{\cdot}01}\cdot\Delta\rho_1 + \frac{c_1''' - c_1'}{0{\cdot}01}\cdot\Delta\rho_3 &= C_1' + \frac{C_1'' - C_1'}{0{\cdot}01}\cdot\Delta\rho_1 + \frac{C_1''' - C_1'}{0{\cdot}01}\cdot\Delta\rho_3 \\ c_3' + \frac{c_3'' - c_1'}{0{\cdot}01}\cdot\Delta\rho_1 + \frac{c_3''' - c_1'}{0{\cdot}01}\cdot\Delta\rho_3 &= C_3' + \frac{C_3'' - C_1'}{0{\cdot}01}\cdot\Delta\rho_1 + \frac{C_3''' - C_1'}{0{\cdot}01}\cdot\Delta\rho_3 \\ 100\{[C_1'' - C_1' - (c_1'' - c_1')]\Delta\rho_1 + [C_1''' - C_1' - (c_1''' - c_1')]\Delta\rho_3\} + C_1' - c_1' &= 0 \\ 100\{[C_3'' - C_1' - (c_3'' - c_1')]\Delta\rho_1 + [C_3''' - C_1' - (c_3''' - c_1')]\Delta\rho_3\} + C_3' - c_1' &= 0 \end{aligned} \qquad (2)$$

The coefficients of $\Delta\rho_1$ and $\Delta\rho_3$ can be regarded as constant and at each trial the last term in equations (2) is altered, the new values of $C_1' - c_1'$ and $C_3' - c_3'$ being substituted. The process is continued until these vanish.

Dynamical Equations.—It is advisable to use tables to facilitate the computation of s, the sector-triangle ratio. The following formulæ will be found very convenient for a machine. The first and third positions of the comet are used as typical, and it is assumed that the angle $2\theta_2$ between the radii vectores from the Sun has been determined. Let

$$j = \tfrac{1}{2}\left(\frac{r_1 + r_3}{2\sqrt{r_1 r_3}\cos\theta_2} - 1\right), \qquad m = \frac{\tau_2^2}{(2\sqrt{r_1 r_3}\cos\theta_2)^3}, \qquad \eta = \frac{m}{\frac{5}{6} + j + \xi} \tag{3}$$

In the first approximation ξ is ignored, and $\log s^2$ is taken from one of the tables. ξ is found from another table* with $\frac{m}{s^2} - j$ as argument, and the new value of η, differing slightly from the first value, is obtained. In the case of moderately short arcs, ξ is very small and can often be ignored.

The same formulæ are used for r_1 and r_2, r_2 and r_3. Let the values of s derived from r_2 and r_3, r_1 and r_3, r_1 and r_2 be s_1, s_2 and s_3 respectively, and let

$$C_1^0 = \frac{t_3 - t_2}{t_3 - t_1}, \qquad C_3^0 = \frac{t_2 - t_1}{t_3 - t_1} \tag{4}$$

then

$$C_1 = \frac{s_2}{s_1} C_1^0, \qquad C_3 = \frac{s_2}{s_3} C_3^0 \tag{5}$$

Differential corrections can be applied to (3), but there is little or no advantage in this, and it is more satisfactory to recompute C_1 and C_3 with the new values of r and θ.

The Formulæ of Gibbs.—In a very interesting paper † the late Dr. R. T. A. Innes advocates the use of these formulæ, and suggests that they may suffice. That they are not adequate for the period of two months is shown by the following figures:—

$C_1 = 0{\cdot}5780220$, $C_3 = 0{\cdot}4885926$; Innes, using the method of Gauss.
$C_1 = 0{\cdot}5781804$, $C_3 = 0{\cdot}4887308$; Innes, using the Gibbs formulæ.
$\Delta C_1 = 0{\cdot}0001584$, $\Delta C_3 = 0{\cdot}0001382$

Substituting these discrepancies in $\Delta\rho_2 = -25{\cdot}2\Delta C_1 + 13{\cdot}0\Delta C_3$, as given on p. 23, we find $\Delta\rho_2 = 0{\cdot}0058$. In other words, the use of the Gibbs formulæ would involve an error of 6 units in the third decimal place. As these formulæ will be used later for a minor planet, some remarks will now be made about them. Let

$$\mu_1 = \tfrac{1}{12}(\tau_2\tau_3 - \tau_1^2), \qquad \mu_2 = \tfrac{1}{12}(\tau_1\tau_3 + \tau_2^2), \qquad \mu_3 = \tfrac{1}{12}(\tau_1\tau_2 - \tau_3^2).$$

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* Gauss, tables ii and iii ; Watson, tables xiii and xiv ; Oppolzer, tables vii and ix ; Bauschinger, tables xix and xx.

† *Memoirs of the British Astronomical Association*, 30, pt. 1, 14–41, 1932.

Then

$$C_1 = C_1^0 \frac{1+\mu_1 r_1^{-3}}{1-\mu_2 r_2^{-3}}, \qquad C_3 = C_3^0 \frac{1+\mu_3 r_3^{-3}}{1-\mu_2 r_2^{-3}} \tag{6}$$

Differentiating these expressions and reducing to their simplest form

$$\Delta C_1 = -3C_1\left(\frac{\mu_1 r_1^{-3}}{1+\mu_1 r_1^{-3}} r_1^{-1} \,.\, \Delta r_1 + \frac{\mu_2 r_2^{-3}}{1-\mu_2 r_2^{-3}} r_2^{-1} \,.\, \Delta r_2\right)$$
$$\Delta C_3 = -3C_3\left(\frac{\mu_3 r_3^{-3}}{1+\mu_3 r_3^{-3}} r_3^{-1} \,.\, \Delta r_3 + \frac{\mu_2 r_2^{-3}}{1-\mu_2 r_2^{-3}} r_2^{-1} \,.\, \Delta r_2\right) \tag{7}$$

The formulæ of Gibbs are more suitable for equal or nearly equal time intervals, and especially for minor planets. Their use for comets is usually limited to much shorter time intervals.

Illustration.—An example to illustrate the application of the method is taken from *Publications of the Lick Observatory*, 7, 440. Prof. A. O. Leuschner uses his own method for computing the orbit of Comet 1909 *e* (Daniel), and the work is reproduced in Prof. R. T. Crawford's *Determination of Orbits of Comets and Asteroids*, pp. 185–192. The first attempt of the present writer to compute an orbit failed completely, due to an error in the printed right ascension of the second position. It is given as 90° 30′ 46″·4 instead of 94° 30′ 46″·4. The observations, together with the direction-cosines and solar co-ordinates (corrected for parallax) referred to the ecliptic, are

G.C.T.	1909 Dec. 7·66050	1909 Dec. 18·75589	1910 Mar. 3·47726
α(1909·0)	94° 9′ 31″·4	94° 30′ 46″·4	111° 14′ 26″·1
δ(1909·0)	+34° 44′ 59″·3	+43° 25′ 59″·3	+ 52° 54′ 37″·2
l	−0·059593	−0·057138	−0·218481
m	+0·978670	+0·937734	+0·833122
n	+0·196605	+0·342621	+0·508108
X	−0·247758	−0·056289	+0·945893
Y	−0·953247	−0·982185	−0·298027
τ^2	1·65216	2·17925	0·03643

$C_1^0 = 0{\cdot}870708$ $\qquad$ $C_3^0 = 0{\cdot}129292$

The equations expressing the geometrical conditions are

$$-0{\cdot}059593c_1\rho_1 + 0{\cdot}057138\rho_2 - 0{\cdot}218481c_3\rho_3 = -0{\cdot}247758c_1 + 0{\cdot}056289 + 0{\cdot}945893c_3$$
$$+0{\cdot}978670c_1\rho_1 - 0{\cdot}937734\rho_2 + 0{\cdot}833122c_3\rho_3 = -0{\cdot}953247c_1 + 0{\cdot}982185 - 0{\cdot}298027c_3$$
$$+0{\cdot}196605c_1\rho_1 - 0{\cdot}342621\rho_2 + 0{\cdot}508108c_3\rho_3 = 0$$

Eliminating ρ_2 between the first and third and also between the second and third gives two equations with $c_1\rho_1$ and $c_3\rho_3$, from which $c_3\rho_3$ and $c_1\rho_1$ are eliminated in turn, yielding

$$c_1\rho_1 = 0{\cdot}14408c_1 + 1{\cdot}35343 - 7{\cdot}67878c_3$$
$$\rho_2 = 2{\cdot}78706c_1 - 0{\cdot}24979 - 12{\cdot}61233c_3$$
$$c_3\rho_3 = 1{\cdot}82359c_1 - 0{\cdot}69213 - 5{\cdot}53335c_3 \tag{8}$$

The equations connecting the heliocentric and geocentric distances are

$$r_1^2 = (\rho_1 + 0{\cdot}918176)^2 + 0{\cdot}127017$$
$$r_2^2 = (\rho_2 + 0{\cdot}917812)^2 + 0{\cdot}125477$$
$$r_3^2 = (\rho_3 + 0{\cdot}454952)^2 + 0{\cdot}776552 \qquad (9)$$

The time intervals are very unequal, and this renders Gibbs's expressions practically useless in this case.

Provisional values $\rho_1 = 0{\cdot}430898$ and $\rho_3 = 1{\cdot}05580$, derived from a preliminary orbit, were used. Substituting these in (8), and solving the first and third for c_1 and c_3,

$$c_1 = 9{\cdot}895539 \qquad c_3 = 0{\cdot}142806 \qquad \rho_2 = 0{\cdot}445024$$

Hence

ρl	$-0{\cdot}025678$	$-0{\cdot}025428$	$-0{\cdot}230672$
X	$-0{\cdot}247758$	$-0{\cdot}056289$	$+0{\cdot}945893$
x	$+0{\cdot}222080$	$+0{\cdot}030861$	$-1{\cdot}176565$
ρm	$+0{\cdot}421707$	$+0{\cdot}417314$	$+0{\cdot}879610$
Y	$-0{\cdot}953347$	$-0{\cdot}982185$	$-0{\cdot}298027$
y	$+1{\cdot}374954$	$+1{\cdot}399499$	$+1{\cdot}177637$
$z = \rho n$	$+0{\cdot}084717$	$+0{\cdot}152475$	$+0{\cdot}536460$
r	1·39535	1·40812	1·74898
2θ	46° 33′ 13″	54° 53′ 51″	8° 20′ 39″

the last line being obtained from

$$\cos 2\theta_2 = \frac{x_1x_3 + y_1y_3 + z_1z_3}{r_1r_3} \text{ etc.}$$

As a check note that $2\theta_2 = 2\theta_1 + 2\theta_3$. This is based on the assumption that the three radii vectores are in a plane, and therefore it checks the accuracy not only of each value of $\cos 2\theta$, but also of ρ_2, which is derived from the fundamental equations on the assumption of the plane conditions being fulfilled. A discrepancy of a few seconds may occur owing to the accumulation of small errors, but if the discrepancy is large the work must be revised.

Differentiating equations (8) and using the provisional values of c_1, c_3, ρ_1 and ρ_3,

$$0{\cdot}892\Delta\rho_1 = -0{\cdot}287\Delta c_1 - 7{\cdot}680\Delta c_3$$
$$0{\cdot}142\Delta\rho_3 = +1{\cdot}823\Delta c_1 - 6{\cdot}590\Delta c_3$$

from which

$$\Delta c_1 = -0{\cdot}373\Delta\rho_1 + 0{\cdot}070\Delta\rho_3$$
$$\Delta c_3 = -0{\cdot}103\Delta\rho_1 - 0{\cdot}003\Delta\rho_3 \qquad (10)$$

Substituting these in the second equation,

$$\Delta\rho_2 = +0{\cdot}260\Delta\rho_1 + 0{\cdot}234\Delta\rho_3$$

These can be used as checks in the subsequent work.

The sector-triangle ratios will now be found from (3) and from these C_1 and C_3 are computed from (4) and (5).

r_1+r_3	3·14433	
$2\sqrt{r_1r_3}\cos\theta_2$	2·77265	
j	0·0670261	
m	0·1022401	
η	0·1135	(provisional value)
$\log s_2^2$	0·0924	
x	0·0156	
ξ	0·0000141	
η	0·1135529	(final value)
$\log s_2^2$	0·0924340	
s_2	1·112287	

$$s_1 = 1{\cdot}079923 \qquad s_3 = 1{\cdot}002212$$
$$C_1 = 0{\cdot}896802 \qquad C_3 = 0{\cdot}143492$$

An increment of $+0{\cdot}01$ is now given to ρ_1, ρ_3 remaining the same; from (6)

$$c_1 = 0{\cdot}891842 \qquad c_3 = 0{\cdot}141783 \qquad \rho_2 = 0{\cdot}447617$$

From the heliocentric distances found from (9)

$$C_1 = 0{\cdot}896060 \qquad C_3 = 0{\cdot}143351$$

Finally, an increment of $+0{\cdot}01$ is given to ρ_3, ρ_1 being unchanged.

$$c_1 = 0{\cdot}896229 \qquad c_3 = 0{\cdot}142780$$
$$C_1 = 0{\cdot}896756 \qquad C_3 = 0{\cdot}143377$$

The geometrical results can obviously be expressed in the form

$$c_1 = 0{\cdot}895539 - 0{\cdot}370\Delta\rho_1 + 0{\cdot}069\Delta\rho_3$$
$$c_3 = 0{\cdot}142806 - 0{\cdot}102\Delta\rho_1 - 0{\cdot}003\Delta\rho_3$$

These figures were obtained by comparing the actual values of c_1 and c_3, and are nearly the same as those found in (10) from the differential corrections. The dynamical results yield

$$C_1 = 0{\cdot}896802 - 0{\cdot}074\Delta\rho_1 - 0{\cdot}005\Delta\rho_3$$
$$C_3 = 0{\cdot}143492 - 0{\cdot}014\Delta\rho_1 - 0{\cdot}011\Delta\rho_3$$

Equating the geometrical and dynamical results

$$+0{\cdot}296\Delta\rho_1 - 0{\cdot}074\Delta\rho_3 + 0{\cdot}001263 = 0$$
$$+0{\cdot}088\Delta\rho_1 - 0{\cdot}008\Delta\rho_3 + 0{\cdot}000686 = 0$$

The absolute term in each case is $C-c$. These equations can be used indefinitely provided the new value of $C-c$ is substituted at each trial.

The solutions are $\Delta\rho_1 = -0{\cdot}010$ and $\Delta\rho_3 = -0{\cdot}022$. The next trial is made with $\rho_1 = 0{\cdot}420898$ and $\rho_3 = 1{\cdot}03380$, from which

$c_1 = 0{\cdot}897730$	$c_3 = 0{\cdot}143893$
$C_1 = 0{\cdot}896772$	$C_3 = 0{\cdot}143900$
$C_1 - c_1 = -0{\cdot}000068$	$C_3 - c_3 = +0{\cdot}000007$
$\Delta\rho_1 = -0{\cdot}00027$	$\Delta\rho_3 = -0{\cdot}0020$
$\rho_1 = 0{\cdot}420628$	$\rho_3 = 1{\cdot}03180$

Light time corrections are now applied.

G.C.T.	Dec. 7·65807	Dec. 18·75337	Mar. 3·47131
τ	1·285305	1·476168	0·190863
τ^2	1·65201	2·179072	0·036429
	$C_1^0 = 0{\cdot}870704$		$C_3^0 = 0{\cdot}129296$.

By the next trial

$$c_1 = 0{\cdot}897692 \qquad c_3 = 0{\cdot}143926 \qquad \rho_2 = 0{\cdot}436894$$
$$C_1 = \ 0{\cdot}897693 \qquad C_3 = \ 0{\cdot}143926$$
$$\Delta\rho_1 = +0{\cdot}000002 \qquad \Delta\rho_3 = +0{\cdot}000020$$
$$\rho_1 = 0{\cdot}420630 \qquad \rho_2 = 0{\cdot}436900 \qquad \rho_3 = 1{\cdot}031820$$

As a check,

$$c_1x_1 + c_3x_3 = x_2, \text{ etc.}$$

x	+0·222691	+0·031325	−1·171326
$c_1x_1 + c_3x_3$		+0·031324	
y	+1·364905	+1·391881	+1·157659
$c_1y_1 + c_3y_3$		+1·391883	
z	+0·082698	+0·149691	+0·524276
$c_1z_1 + c_3z_3$		+0·149694	

If $\Delta\rho_1$ and $\Delta\rho_3$ are the corrections required to produce agreement, then, since

$$\Delta x = l \,.\, \Delta\rho \qquad \Delta y = m \,.\, \Delta\rho \qquad \Delta z = n \,.\, \Delta\rho$$

it follows that

$$c_1l_1 \,.\, \Delta\rho_1 + c_3l_3\Delta\rho_3 = x_2 - (c_1x_1 + c_3x_3), \text{ etc.}$$

Hence, in units of the sixth decimal

$$-0{\cdot}053\Delta\rho_1 - 0{\cdot}031\Delta\rho_3 = +1$$
$$+0{\cdot}878\Delta\rho_1 + 0{\cdot}120\Delta\rho_3 = -2$$
$$+0{\cdot}176\Delta\rho_1 + 0{\cdot}073\Delta\rho_3 = -3$$

The first two yield $\Delta\rho_1 = +0{\cdot}000003$ and $\Delta\rho_3 = -0{\cdot}000040$, which nearly agree with the third. Hence the final values are

$$\rho_1 = 0{\cdot}420633 \qquad \rho_3 = 1{\cdot}031780$$
$$r_1 = 1{\cdot}38542 \qquad r_3 = 1{\cdot}72827$$

The reader may inquire why the values of ρ found by equating the geometrical and dynamical values c and C are subsequently altered by applying geometrical considerations only. The object of this is to reduce the middle-place residuals; the coefficients of $\Delta\rho$ in the equations for C are so small that the alterations produced in C are insignificant.

The elements, computed by the method * already described by the present writer, together with those found by Leuschner, are

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* *Memoirs of the British Astronomical Association*, 30, pt. 1, especially pp. 54–55.

	Present Method		Leuschner	
		Epoch	1909 Dec. 18·75330	
T	1909 Nov. 28·70594	M	3° 02′ 46″·9	
	° ′ ″			
ω	3 27 41·0	ω	3 28 42·6	1909·0
Ω	70 59 03·3	Ω	70 58 54·4	
i	19 26 59·6	i	19 26 48·0	
e	0·602644	e	0·602481	
a	3·47719	a	3·47587	
μ	547″·223	μ	547″·5362	

Residuals

Middle place $\begin{cases}\Delta\lambda + 0''\cdot5 \\ \Delta\beta - 1''\cdot5\end{cases}$ First place $\Delta\alpha + 1''\cdot0$ $\Delta\delta + 0''\cdot2$

Third place $\Delta\alpha + 0''\cdot4$ $\Delta\delta + 0''\cdot5$

Application to Minor Planets.—The application of the formulæ of Gibbs with the differential corrections will now be shown for the case of minor planet 1107 (1929 FB). The observational data are taken from the work of Dr. Raffaele Invrea,* and his values for the direction-cosines and equatorial solar co-ordinates are used. The following will be necessary in the computation :—

$$\begin{aligned} c_1\rho_1 &= -\ 5\cdot03416c_1 + 9\cdot46582 - 11\cdot62976c_3 \\ \rho_2 &= -15\cdot28966c_1 + 18\cdot67529 - 17\cdot57093c_3 \\ c_3\rho_3 &= -11\cdot24356c_1 + 10\cdot24490 - 6\cdot77466c_3 \end{aligned} \qquad (11)$$

Provisional values $\rho_1 = 2\cdot1596$ and $\rho_3 = 2\cdot3472$ were adopted, yielding

$$c_1 = 0\cdot5035335 \qquad c_3 = 0\cdot5024624 \qquad \rho_2 = 2\cdot147705$$

The equations connecting ρ and r are

$$\begin{aligned} r_1^2 &= \rho_1^2 + 1\cdot813084\rho_1 + 0\cdot990503 \\ r_2^2 &= \rho_2^2 + 1\cdot934271\rho_2 + 1\cdot006946 \\ r_3^2 &= \rho_3^2 + 1\cdot438675\rho_3 + 1\cdot021750 \end{aligned} \qquad (12)$$

Differentiating these and substituting the provisional values of ρ and r,

$$\Delta r_1 = 0\cdot991\Delta\rho_1 \qquad \Delta r_2 = 0\cdot996\Delta\rho_2 \qquad \Delta r_3 = 0\cdot974\Delta\rho_3$$

Differentiating (11) and substituting the provisional values of ρ and c,

$$\begin{aligned} \Delta\rho_1 &= -14\cdot30\Delta c_1 - 23\cdot12\Delta c_3 \\ \Delta\rho_3 &= -22\cdot41\Delta c_1 - 18\cdot19\Delta c_3 \end{aligned}$$

from which

$$\begin{aligned} \Delta c_1 &= +\ 0\cdot0705\Delta\rho_1 - 0\cdot0896\Delta\rho_3 \\ \Delta c_3 &= -\ 0\cdot0869\Delta\rho_1 + 0\cdot0554\Delta\rho_3 \\ \Delta\rho_2 &= -15\cdot29\Delta c_1 - 17\cdot57\Delta c_3 \\ &= \quad 0\cdot4488\Delta\rho_1 + 0\cdot3966\Delta\rho_3 \end{aligned}$$

* "Sulla determinazione di un' orbita ellittica per mezzo della macchina calcolatrice," in *Memorie della Società Astronomica Italiana*, vol. vi, No. 8, 1932.

For the dynamical equations the following constants are used :—

$$C_1^0 = 0{\cdot}500529 \qquad C_3^0 = 0{\cdot}499471$$
$$\mu_1 = 0{\cdot}0205786 \qquad \mu_2 = 0{\cdot}1033304 \qquad \mu_3 = 0{\cdot}0207536$$

From the provisional values of ρ

r	3·09354	3·12631	3·14769
$\frac{\mu_3}{r}$	0·0006951	0·0033817	0·0006654

$$C_1 = 0{\cdot}5025767 \qquad C_3 = 0{\cdot}5014993 \qquad \text{(from (6))}$$

The differential corrections found from (7) are

$$\Delta C_1 = -0{\cdot}00034\Delta r_1 - 0{\cdot}00163\Delta r_2$$
$$\Delta C_3 = -0{\cdot}00032\Delta r_3 - 0{\cdot}00163\Delta r_2$$

Replacing Δr_1, Δr_2 and Δr_3 by their expressions in terms of $\Delta\rho_1$ and $\Delta\rho_3$, and noting that

$$\Delta r_2 = 0{\cdot}996\Delta\rho_2 = 0{\cdot}4470\Delta\rho_1 + 0{\cdot}3950\Delta\rho_3$$

we find

$$\Delta C_1 = -0{\cdot}0011\Delta\rho_1 - 0{\cdot}0006\Delta\rho_3$$
$$\Delta C_3 = -0{\cdot}0007\Delta\rho_1 - 0{\cdot}009\Delta\rho_3$$

Equating the geometrical and dynamical results

$$-0{\cdot}0716\Delta\rho_1 + 0{\cdot}0890\Delta\rho_3 - 0{\cdot}0009568 = 0$$
$$+0{\cdot}0862\Delta\rho_1 - 0{\cdot}0563\Delta\rho_3 - 0{\cdot}0009631 = 0$$
$$\Delta\rho_1 = +0{\cdot}0383 \qquad \Delta\rho_3 = +0{\cdot}0416$$

The next trial is made with $\rho_1 = 2{\cdot}1979$ and $\rho_3 = 2{\cdot}3888$.

$$c_1 = 0{\cdot}5025036 \qquad c_3 = 0{\cdot}5014446 \qquad \rho_2 = 2{\cdot}181333$$
$$C_1 = 0{\cdot}5025011 \qquad C_3 = 0{\cdot}5014359$$
$$-0{\cdot}0716\Delta\rho_1 + 0{\cdot}0890\Delta\rho_3 - 0{\cdot}0000025 = 0$$
$$+0{\cdot}0862\Delta\rho_1 - 0{\cdot}0563\Delta\rho_3 - 0{\cdot}0000087 = 0$$
$$\Delta\rho_1 = +0{\cdot}000225 \qquad \Delta\rho_3 = +0{\cdot}000230$$

No further corrections can be made by Gibbs's formulæ, but it is desirable to check the results by the exact equations (3).

$$\rho_1 = 2{\cdot}198125 \qquad \rho_3 = 2{\cdot}389030$$
$$c_1 = 0{\cdot}5024991 \qquad c_3 = 0{\cdot}5014378 \qquad \rho_2 = 2{\cdot}181522$$
$$C_1 = 0{\cdot}5025096 \qquad C_3 = 0{\cdot}5014324$$
$$\Delta\rho_1 = +0{\cdot}000031 \qquad \Delta\rho_3 = -0{\cdot}000091$$
$$\rho_1 = 2{\cdot}198156 \qquad \rho_3 = 2{\cdot}388939$$

These values cannot be considered more exact than the previous ones ; the Gibbs expressions are sufficient in this case.

The results are

$\rho_1 = 2{\cdot}19816$	$\rho_3 = 2{\cdot}38894$	Present method
$\rho_1 = 2{\cdot}19803$	$\rho_3 = 2{\cdot}38882$	Invrea

The method for reducing the residuals in the middle place is the same as that previously described.

It may be noticed that the coefficients for $\Delta\rho$ in the equations for C are very small compared with those for c, and the corrections could be made by using the geometrical results above. This applies to all cases where the heliocentric distances are large.

THE SPOT ON SATURN.

W. T. Hay.

On 1933 August 3 at $22^h\ 35^m$ G.C.T., while observing Saturn in my 6-inch Cooke refractor, I noticed, on the equatorial zone of the planet, a large bright spot which had just crossed the central meridian of the disc.

The existence of this spot was confirmed a few minutes after discovery by Dr. W. H. Steavenson, to whom I had telephoned informing him of the unusual aspect of the planet.

The spot, which was elliptical in outline, appeared to be much brighter than the remainder of the equatorial zone, and the preceding and following ends were well defined. It extended about one-fifth of the planet's diameter in longitude, and occupied the entire visible width of the equatorial zone in latitude.

By August 6 it appeared to be lengthening in longitude, and subsequent observations showed a continuance of this extension in longitude.

The preceding end later became indistinct, and reliable records of transits of the spot across the central meridian of the disc were only to be obtained by observing the following end, which remained well defined throughout the period covered by my observations (August 3 to August 18).

On August 9 it was noted that the portion of the equatorial zone immediately following the spot was so dusky as to be indistinguishable in tone from the dark belt immediately north of it, while the bright spot itself extended slightly northwards into this belt.

Following are my estimated times (G.C.T.) of transit of the spot across the central meridian of the disc :—

Date	Preceding End	Centre of Spot	Following End
	h m	h m	h m
1933 Aug. 6	...	22 12	...
,, ,, 9	1 7	1 34	1 58
,, ,, 9	...	22 5	22 31
,, ,, 12	...	21 48	22 10
,, ,, 18	...	...	21 50

THE ROTATION PERIOD OF THE EQUATORIAL ZONE OF SATURN, FROM OBSERVATIONS OF THE WHITE SPOT, 1933 AUGUST 3–SEPTEMBER 16.

Rev. J. P. Rowland, S.J.

On 1933 August 3 Mr. W. Hay, observing with a 6-in. refractor at Norbury, detected a conspicuous white spot on the equatorial zone of Saturn, which he judged had just crossed the central meridian at $22^h\ 30^m$ G.C.T. On August 4 the British Astronomical Association issued a circular, notifying observers of the occurrence, but as this did not reach the writer till the morning of August 7 there was no opportunity of observing the spot till August 9, when it was observed with the 15-in. refractor of Stonyhurst College Observatory, and was seen to be on the following side of the central meridian at $21^h\ 30^m$. Observation was much interrupted by clouds, but the spot was estimated to be on the central meridian, as nearly as could be judged, at $22^h\ 00^m$ G.C.T. At $22^h\ 10^m$ it was considered to be definitely past the central meridian. Clouds then prevented further observation till $23^h\ 10^m$, when the spot was again seen at about one-third of the planet's diameter from the preceding limb. The spot was of elliptical shape and was estimated to extend in longitude about one-third of the planet's equatorial diameter, and in latitude from the projection of the crepe ring to the north tropical belt, and it was noted that the equatorial belt was notably brighter on the preceding than on the following side of the spot. A drawing was made showing the appearance when the spot was on the central meridian.

From the above observation, assuming an approximate rotation period of $10\frac{1}{4}$ hours, a table of approximate central meridian passages for every rotation to the end of the month was calculated, and a watch was kept on every occasion when the spot would be on the face of the planet, but clouds prevented its being seen again till August 23. With a view to obtaining more accurate values of the time of central meridian transit than could be made by simple eye estimates, it had been decided to make series of measurements with a filar micrometer of excellent quality made by Troughton & Sims, and reconditioned about two years ago by Messrs. Cooke of York. The screw pitch of this instrument is 0·02 in., and its scale value when used with the 15-in. refractor, of 15 ft. 5 in. focal length, is $22''{\cdot}3033$ per turn of the heads, or $0''{\cdot}223$ per division. It was realised that owing to instability of the image from atmospheric causes, and the diffuse character of the object to be measured, an individual setting would be liable to a considerable margin of error, and so no attempt was made to measure the actual time of central meridian transit, by setting the wires at a distance apart equal to the semi-diameter of the planet, and then with one wire set to the limb observing the time of transit over the other; the method adopted was to make as many measurements as possible of the time and position of the

spot as it passed across the disc. In this way it was hoped to eliminate the errors of individual observations.

The procedure adopted was, after setting the horizontal wire to the major axis of the rings, to bring one limb of the planet to one of the wires, which was set to zero, bring the other wire to the centre of the spot, note the time, and then read the micrometer, and to repeat this process as often as possible as the spot passed across about the middle third of the planet's disc. The values so obtained being plotted against each other, with the micrometer readings as ordinates and the times as abscissæ, and the best straight line drawn through the points so obtained, the time at which this line had the ordinate corresponding to the semi-diameter of the planet gave the time of transit over the central meridian.

Observations by this method were made on August 23, 26 and 29, and on September 1 and 16, but with slight modifications. On August 23 and 26 all measurements were made from the preceding limb, but on August 29 and September 1 measurements were made alternately from both limbs, and thus two lines of transit were obtained which must intersect on the central meridian; thus an additional check was obtained on the position at which they must be drawn. A further check is found in the consideration that they must be equally inclined to the axes of co-ordinates. In this way it was hoped to eliminate systematic error which might arise from always setting to one limb. Observing conditions were not good on any of the nights, definition being but poor to fair, and at times bad, which latter condition prevailed throughout the observations on September 1 and 16; on the last of these dates the spot was so indistinct that it was not thought worth while proceeding with the observation, and only a few settings were made, and these all from the preceding limb. The results on these dates are considered of inferior value, and have not been used in determining the rotation period, but are included for completeness, and their residuals given as referred to the more reliable observations.

It should be noted that in all these later observations the character of the spot was greatly changed from its appearance on August 9, its preceding end merging into the general brightness of the equatorial belt, and there might be question of the possibility of identifying its centre. There was, however, a distinct enlargement near the following end, which could be definitely identified till the last date of observation, and it was the centre of this enlargement which was taken in all cases as the point of measurement. The numbers of settings obtained on August 23, 26 and 29, from which the rotation period has been deduced, were 15, 29 and 48 respectively, the relatively small number on August 23 being due to frequent interruptions by clouds.

The following values of the equatorial diameter of the planet in terms of divisions of the micrometer head were obtained on August 26, and may serve as an indication of the degree of accuracy attainable :—

Equatorial Diameter—82·5, 80·5, 81·0, 78·0, 81·0, 83·0, 82·5, 82·5, 81·5, 83·0. Mean 81·55.

Equatorial Semidiameter—40·8 divisions = 9″·1.

It may be noted that as an error of five divisions of the micrometer head, or one-thousandth of an inch, in the setting of the wires would correspond to an error of about a quarter of an hour in time, the margin of error from a single observation is liable to be considerable, and the same applies with equal, or greater, force to simple eye estimates of the time when the spot is on the central meridian. By taking a large number of measurements, however, the errors, which are as liable to be positive as negative, largely balance each other and are possibly eliminated.

The times of central meridian passage and the rotation periods deduced from successive pairs of all the Stonyhurst observations, with the numbers of rotations on which they are based, are exhibited in the following table :—

TABLE I

Date	Central Meridian Passage	Rotations	Period
1933	h m		h m
Aug. 9	22 00	...	... ...
23	23 46	33	10 14·12
26	23 25	7	10 14·14
29	23 04	7	10 14·14
Sept. 1	22 39	7	10 13·57
16	20 57	35	10 14·23

The previously accepted value of the rotation period of the equatorial zone, determined by Professor Asaph Hall in 1876, is $10^h\ 14^m{\cdot}4$. The present spot, owing to its rapid change in character, especially during its early life, presents some difficulty in determining a consistent value of the rotation period, which appeared to shorten from about $10^h\ 16^m$ or $10^h\ 17^m$ in the first days of observation to about $10^h\ 13^m$ on September 10, with mean values for 30 rotations from August 6 to 18 of $10^h\ 15^m{\cdot}4$, and from August 9 to September 10 of $10^h\ 14^m{\cdot}25$ with a sudden discontinuity in the residuals about August 23 (*The Observatory*, September and October). As all the writer's observations, with the exception of the eye estimate on August 9, are subsequent to this, there may be some doubt as to the rotation period deduced from them being applicable to the earlier period. The observations of August 23, 26 and 29 are considered, however, to give a reliable value of the rotation period during that interval, and with a view to the correlation of all the observations available, an ephemeris was worked out for every rotation from August 3 to September 16, which covers the whole period of observations to which the writer has had access.

A considerable number of observations have been collected from various sources, and are given in Table II, with residuals referred to the time of transit on August 23, $23^h\ 46^m$, and rotation period $10^h\ 14^m{\cdot}14$. The sources of information are acknowledged as follows :—

A = *L'Astronomie*, 1933 September.

H = *Harvard Announcement Cards*, Nos. 273, 274, 277.

N = *Astronomische Nachrichten*, Nos. 5968, 5970, 5973, 5975.

P = Personal Communication.

TABLE II

Date	Observer	Rotations	G.C.T.	O − C
1933			h m	m
Aug. 3.	W. Hay, Norbury (A)	0	22 30	−11·4
3.	A. Weber, Berlin (N)	0	22 30	−11·4
5.	Hellweg, U.S. Nav. Obs. (A)	3	5 18	− 5·9
6.	A. Weber, Berlin (N)	5	1 39·4	−12·7
6.	J. E. Willis, U.S. Nav. Obs. (H)	5	1 44·5	− 7·6
6.	Mme. Flammarion, Juvisy (A)	7	22 10	−10·4
6.	W. Hay, Norbury (P)	7	22 12	− 8·4
6.	A. Weber, Berlin (N)	7	22 12·5	− 7·9
6.	W. Strachan, Bournemouth (P)	7	22 16·5	− 3·9
6.	W. Münch, Potsdam (N)	7	22 20	− 0·4
7.	J. E. Willis, U.S. Nav. Obs. (H)	8	8 35	− 0·4
9.	J. Korn, Zweibrücken (N)	12	1 30	− 1·1
9.	W. Hay, Norbury (P)	12	1 34	+ 2·9
9.	A. Weber, Berlin (N)	14	21 56·5	− 2·9
9.	J. P. Rowland, Stonyhurst	14	22 00	+ 0·6
9.	W. Hay, Norbury (P)	14	22 05	+ 5·6
9.	C. Fedtke, Königsberg (N)	14	22 05	+ 5·6
9.	W. Strachan, Bournemouth (P)	14	22 06·5	+ 7·1
9.	Mme. Flammarion, Juvisy (A)	14	22 08	+ 8·6
9.	P. Emanuelli, Rome (A)	14	22 10	+10·6
11.	G. Van Biesbroek, Yerkes (H)	17	4 43·7	+ 1·9
12.	W. Hay, Norbury (P)	21	21 48	+ 9·6
23.	J. P. Rowland, Stonyhurst	47	23 46	0·0
26.	J. P. Rowland, Stonyhurst	54	23 25	0·0
26.	P. H. Fauth, Grünwald (N)	54	23 36	+11·0
27.	P. H. Fauth, Grünwald (N)	(56	20 35	+41·7?)
29.	J. P. Rowland, Stonyhurst	61	23 04	0·0
30.	W. Strachan, Bournemouth (P)	63	19 30	− 2·3
Sept. 1.	J. P. Rowland, Stonyhurst	68	22 39	− 4·0
16.	J. P. Rowland, Stonyhurst	103	20 57	− 0·9

A striking feature of the above table is the predominance of negative residuals, some rather large, till August 9, and of positive ones, also running to large values, on that day. They indicate a definitely longer rotation period as deduced from the observations during that interval than that determined by the writer. The discrepancies, however, between the times given by different observers for the same transit leave it in doubt as to what should be taken as the true time of transit for any given rotation. All the observations quoted refer to the centre of the spot, and it is evident that no uniform value of the rotation period during the whole interval under review could be reconciled with the data, even if the times of observation of individual rotations be averaged. Differences of 10^m between the extreme values of the time of transit at the 7th rotation, and of $13^m.5$ at the 14th, can only be attributed to differences of opinion as to the time when the centre of the spot was on the central meridian of the planet, but, as indicated above, in the opinion of the writer, this amount does not exceed the probable error of a single observation, whether a simple eye estimate or a micrometer measure. The change from predominantly negative to positive residuals

may perhaps be attributable to the changing character of the spot, and the difficulty of assigning the centre of an unsymmetrical object. It may, however, be indicative of a physical change in progress, and a real difference in the rotation period of the matter forming the spot. If this be so, an explanation may perhaps be found in supposing that the matter was rising to higher levels in the planet's atmosphere, where the velocity of translation would be expected to be greater, and so there would be an apparent acceleration of the rotation of the planet, or, more strictly, of its atmosphere, which is all that can be observed. The rapid extension of brightness of the equatorial zone on the preceding side of the spot gives support to this suggestion, and it is only necessary to assume that above the opaque cloudy surface of the planet's atmosphere which we normally see there is an outer transparent layer, analogous to the stratosphere of the earth, having a higher velocity of rotation, into which the matter of the spot could rise. This would also account for the following end, and centre, of the spot giving a slower rate of rotation in the early days, as both would be at a lower level than in the later stages, when the primary cause of the outburst having ceased to operate, the whole of the matter of the spot would have risen to its highest and final level. It may be that this stage was reached about August 23, but the occurrence of some very small residuals as early as the 6th seems to point to the possibility that the discrepancies in the times of the early observations may be due mainly to differences in the estimated centre of the spot.

In addition to the spot discovered by Mr. Hay on August 3, a number of other spots were announced to have made their appearance subsequently, and in the course of this investigation it became apparent that some of the observations of transit times which were supposed to be of the original spot were in reality of one or other of these subsidiary spots. Thus the residual of $+41^{m}.7$ of the observation of P. H. Fauth on August 27 seemed to preclude this observation from being, as supposed, of the centre of the original spot, but no explanation was forthcoming till it was compared with an observation quoted by B. M. Peek in the October number of *The Observatory* (p. 311). It is there stated that "the following end of a bright portion of the equatorial zone, which seems to be identical with the following boundary of the white spot," was observed to be central on September 10 at $22^{h}\ 40^{m}$. This is just one hour later than the transit time of the centre of the spot calculated as in Table II, and if 18^{m} be allowed as the interval between the transits of the centre and following end of this object, a rotation period over 33 rotations is obtained in conformity with that given in this paper. It appears probable, therefore, that these two observations relate to the same feature, but not to that measured by the writer on August 23, 26 and 29.

A more interesting correlation is found on comparing two observations of Mr. Strachan with the transit time of a new spot discovered at the Lowell Observatory on August 29, as reported in *Harvard Announcement Card* No. 279. The observations with the resulting rotation periods are as follows :—

Date	Observer	Rotations	G.C.T.	Period
			h m	h m
Aug. 29.	Lowell Obs. (H)	0	4 57	... ...
Sept. 8.	W. Strachan (P)	25	20 44·25	10 13·89
11.	W. Strachan (P)	32	20 33	10 14·25
				Mean 10 14·07

The above mean, differing by only $0^m.07$ from that deduced from the writer's observations of the original spot, affords a striking and independent confirmation of its probable accuracy. An observation at Headley on September 14, quoted in the October number of *The Observatory*, gives the central meridian transit on that date as $20^h\ 24^m$, and this combined with the Flagstaff observation gives a rotation period over 39 rotations of $10^h\ 14^m.54$, and with the above values a mean for all the observations of this object of $10^h\ 14^m.23$.

The results put forward in this paper have not been corrected for differences in the times of light transmission, or for the change in geocentric longitude of the planet, and it is recognised that the observations are few on which to base a final value, which must await the publication of a more complete list of observations. It may be that no absolute value of the rotation period of any zone of this planet can be assigned, as it may not be constant, and observation of different markings, which may be at different levels, may be expected to give different periods. In view, however, of the considerable discrepancies in the times assigned by different observers for the same transits, it seems desirable that observers, in publishing their results, should indicate the method of observation, so that it may be judged what weight should be assigned to them in any combined scheme of reduction.

Stonyhurst College Observatory :
1933 *October* 23.

OCCULTATIONS OF STARS BY THE MOON, OBSERVED AT THE UNIVERSITY OF LONDON OBSERVATORY.

(Communicated by the Director)

This report, concerning occultations observed at the University of London Observatory, is in continuation of two printed in *M.N.*, **91** (1931) and **93** (1933). "Eye and ear" is the method of observation adopted by G, but a stop-watch is used by R. The three co-ordinates of the 24-inch reflector are given year by year in the *N.A.*, and the 6-inch refractor is the guide for the reflector; the 8-inch Cooke and the $4\frac{1}{2}$-inch Newtonian are within a negligible distance (17^m) of the main instrument. Column 6 in Table I contains the actual time of observation. For the purpose of interpolating for parallax, right ascension and declination (but for no other purpose) $0^h.0030$ was added to the G.C.T. in the case of serial numbers 19 to 22 and $0^h.00253$ in the remaining serial numbers, as suggested by Professor E. W. Brown. No other correction for lag, personal error, etc., has been included. The reductions were made by R and checked by G.

TABLE I

Observations

Serial No.	Star	Mag.	Phase	Date	G.C.T.	Telescope	Observer
				1932	h m s		
19	*i* Aqu	4·4	D	Dec. 3	18 26 48·88	R	R
20	B.D. − 14° 6218	6·9	D	3	20 9 48·97	C	G
21	*ρ* Leo	3·8	D	19	0 24 4·11	R	R
22	*ρ* Leo	3·8	R	19	1 28 52·31	R	R
				1933			
23	W.Z.C. 176	6·7	D	Feb. 2	22 50 11·94	R	R
24	47 B Aur	6·0	D	Mar. 4	19 56 42·90	R	R
25	5 B Can	6·4	D	7	23 57 3·70	R	R
26	W.Z.C. 663	6·7	D	May 3	22 31 40·35	R	R
27	79 Leo	5·5	D	5	20 56 2·05	R	R
28	*ϵ* Ari	4·6	R	Aug. 12	22 49 40·75	N	R
29	*μ* Cap	5·2	D	Sept. 3	2 4 57·46	G	G
30	W.Z.C. 495	6·9	R	14	2 31 46·91	R	R
31	9 Can	6·2	R	15	3 45 23·02	R	R
32	W.Z.C. 554	7·2	R	15	3 57 55·24	R	R
33	*η* Can	5·5	R	Oct. 13	1 0 50·32	R	R
34	W.Z.C. 587	6·7	R	13	5 2 5·42	R	R
35	39 Can	6·5	R	13	5 5 1·93	R	R
36	40 Can	6·5	R	13	5 11 43·42	R	R
37	B.D. 20° 2175	7·7	R	13	5 42 30·88	R	R
38	40 Can	6·5	R	13	5 11 44·02	C	G

TABLE II

Reductions

No.	*χ*	*ρ*	*σ′* − *σ*	*χ* − *ρ*	cos (*χ* − *ρ*
	° ′	° ′		° ′	
19	85 38	64 50	−2·1	+ 20 48	+0·935
20	54 1	64 37	−3·1	− 10 36	+0·983
21	101 29	116 26	+4·2	− 14 57	+0·966
22	223 25	116 31	+2·2	+106 54	−0·291
23	85 37	71 32	−1·5	+ 14 5	+0·970
24	12 49	85 38	+2·4	− 72 49	+0·295
25	83 30	105 17	+1·5	− 21 47	+0·929
26	146 11	114 31	−0·6	+ 31 40	+0·851
27	201 33	118 0	+1·5	83 33	+0·112
28	208 31	72 5	+2·3	136 26	−0·725
29	356 14	65 0	(+6·7)	291 14	+0·362
30	341 9	100 22	−0·1	240 47	−0·488
31	298 2	106 5	+1·4	191 57	−0·978
32	260 40	106 7	+2·0	154 33	−0·903
33	295 40	108 37	+1·5	187 3	−0·992
34	258 11	109 16	−0·4	148 55	−0·856
35	316 52	109 17	+1·7	207 35	−0·886
36	310 31	109 18	+1·0	201 13	−0·932
37	234 17	109 23	+1·2	124 54	−0·572
38	310 31	109 20	+1·3	201 11	−0·932

In six instances the *N.A.* gave no co-ordinates, and serial No. 37 was not predicted. Table III gives the necessary particulars for these seven stars.

TABLE III

Serial No.	R.A. 1932·0	Reduction to Date	Dec. 1932·0	Reduction to Date	Authority
		s		"	
20	22 5 7·61	+3·09	−13 38 1·9	+11·6	E. W. Brown.
	R.A. 1933·0		Dec. 1933·0		
23	2 55 2·29	+0·73	+21 21 3·3	+11·4	E. W. Brown.
26	9 47 24·60	+1·87	13 22 47·6	− 4·5	Dirk Brouwer.
30	7 8 42·19	+2·72	25 51 43·3	+ 2·5	Dirk Brouwer.
32	8 2 37·19	+2·42	22 39 1·5	− 0·2	Dirk Brouwer.
34	8 35 51·97	+2·99	20 0 57·9	− 4·7	Cat. Zodiacal Stars, Washington.
37	8 37 5·93	+2·98	19 49 10·5	− 4·7	Abbadia 14263, No. 5149.

Observers:

C. C. L. Gregory (G).
Frank Robbins (R).

Telescopes:

4½-inch reflector (N).
6-inch refractor (G).
8-inch refractor (R).
24-inch reflector (Cassegrain) (C).

Serial No.	*Notes*
21.	Much mist around Moon—not easy.
24.	Through thin clouds.
29.	Difficult. R using N failed to find star.
37.	Unpredicted. Found by Cordoba Photographs.

Errata

M.N., **93**, 668, 669, Professor Milne's paper on "World Structure": transpose fourth footnote (§) on p. 668 with first footnote (*) on p. 669.

M.N., **93**, 676, *for* re-shifts *read* red-shifts.